The paper is divided into two sections. There are four questions in each section.

Answer TWO questions from section A, and TWO questions from section B. All questions carry the same weight.
SECTION A: Answer TWO questions from this section.)

Question 1.

(a) Give the definitions of a forward and of a futures contract maturing at $T_1$, and derive an expression for the forward price at $t$ of a risky asset with price $S_t$ at $t$, assuming a constant risk-free rate of $r$. [4 points]

(b) Assume a futures contract maturing at $T_1$ is negotiated at a price of $f_t$ whose evolution is modelled by a Geometric Brownian motion

$$df_t = \mu f_t dt + \sigma f_t dW_t. \tag{1}$$

Consider a European call maturing at $T < T_1$ written on the future which allows you to enter into a future contract at time $T$ at future price of $K$ instead of $f_T$.

(i) Explain why this is equivalent to a cash payment of $\max(f_T - K, 0)$ at $T$. [2 points]

(ii) If we let $V(f, t)$ be the price of this call, as function of the futures price $f$ and of time, derive the Partial Differential Equation which $V$ should satisfy prior to $T$. [3 points]

(c) (i) Show that for a European option on our future paying off $G(f_T)$ at $T$, the price at $t$ can be expressed as

$$V(f, t) = e^{-rT} \int_{-\infty}^{\infty} G\left(f e^{-\frac{1}{2} \sigma^2 \tau + \sigma \sqrt{\tau} z}\right) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}. \tag{2}$$

[4 points]

(ii) Use this to derive Black’s formula for a European call on a future. [4 points]

(d) Suppose you have sold the call. Compute the number of future contracts you should enter into at the $t$ in order to hedge your sale. [3 points]
Solution:

(a) **Forward**: At $T_1$, exchange one unit of the risky asset against an amount of $F_{t,T_1}$, agreed upon at $t$;

future: taking a long position in the future at $t$ at future price $f_t$ is costless and generates a profit/loss of $f_{t+dt} - f_t$ at $t + dt$ (in the limit of continuous settlement); at maturity $T_1$, $f_{T_1} = S_{T_1}$.

By either using risk-neutral pricing or an elementary AoA argument one shows: $F_{t,T_1} = e^{-r(T_1-t)}S_t$, $t \leq T_1$. (Students have seen both arguments.)

(b) (a) (i) If you hold the call on the future, you can exercise it at $T$ to enter into a future contract at a future price of $K$ instead of $f_{T_1}$, which you will do off $f_T > K$. At $T + dt$, this will then give you a profit of $f_{T+dt} - K$, instead of $f_{T+dt} - f_T$, that is, a profit of $f_T - K$.

(ii) Usual hedging argument: at $t < T$, hedge a short position in the derivative by taking a long position of $\Delta_t$ in the future. Since this is cost-less, the value of the hedged portfolio at $t$ is

$$\Pi_t = V(f_t, T).$$

At $t + dt$, the value of the portfolio is

$$\Pi_{t+dt} = V(f_t, t) + dV(f_t, t) + \Delta_t df_t.$$

Ito’s lemma shows that the portfolio is risk-free over $[t, t + dt]$ if $\Delta_t = \partial_t V(f_t, t)$, with $d\Pi_t = \partial_t V + \frac{1}{2}\sigma^2 f_t^2 \partial^2_t V$ (derivatives evaluated in $(f_t, t)$). By AoA, the return should then be the risk-free return, $r$: $d\Pi_t = r\Pi_t$, or

$$\partial_t V + \frac{1}{2}\sigma^2 f_t^2 \partial^2_t V = rV.$$

(c) (i) The final-value problem for the European option we have to solve is:

$$\begin{cases} \partial_t V + \frac{1}{2}\sigma^2 f_t^2 \partial^2_t V = rV, & t < T \\ V(f, T) = G(f), & t = T. \end{cases}$$

By the Feynman-Kac theorem,

$$V(f, t) = e^{-r(T-t)}\mathbb{E}\left(G(\hat{f}_t) \mid \hat{f}_t = f\right),$$

with $d\hat{f}_t = \sigma \hat{f}_t dW_t$. Solving the SDE from $t$ to $T$ with initial value $\hat{f}_t = f$ at $t$ gives

$$\hat{f}_T = f e^{-\frac{1}{2}\sigma^2(T-t) + \sigma(W_T - W_t)}.$$

Plugging this into the Feynman - Kac formula and using that $W_T - W_t = \sqrt{T-t}Z$ in distribution, with $Z \sim N(0, 1)$ then yields the stated formula.
(ii) We have to compute

\[ e^{-r\tau} \int_{-\infty}^{\infty} \max \left( fe^{-\frac{1}{2}\sigma^2 r + \sigma \sqrt{\tau} z} - K, 0 \right) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}. \]

The integration extends over \( \{z : z > -d_-\} \) where \( d_- := (\log(f/K) - \frac{1}{2}\sigma^2 r)/\sigma \sqrt{\tau} \), so

\[ V(f, t) = e^{-r\tau} \int_{-d_-}^{\infty} fe^{-\frac{1}{2}\sigma^2 r + \sigma \sqrt{\tau} z} - \frac{1}{2} z^2 e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} - e^{-r\tau} K \int_{-d_-}^{\infty} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \]

\[ = e^{-r\tau} f \int_{-d_-}^{\infty} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} e^{-r\tau} K \int_{-d_-}^{\infty} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \]

\[ = f e^{-r\tau} \Phi(d_+) - Ke^{-r\tau} \Phi(d_-), \]

where we used that \( \frac{1}{2}\sigma^2 r - \sigma \sqrt{\tau} z + \frac{1}{2} = z^2(z - \sigma \sqrt{\tau})^2 \), and where \( d_+ := d_- + \sigma \sqrt{\tau} \) and \( \Phi \) is the cdf of \( Z \sim N(0, 1) \).

(d) We should enter into a long position of \( \Delta(f, t) := \partial_f V(f, t) \) futures. Differentiating under the integral sign in (c):

\[ \partial_f V(f, t) = e^{-r\tau} \int_{-\infty}^{\infty} G'(fe^{-\frac{1}{2}\sigma^2 r + \sigma \sqrt{\tau} z}) e^{-\frac{1}{2}\sigma^2 r + \sigma \sqrt{\tau} z} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}. \]

For a call,

\[ G'(f) = \begin{cases} 1, & f > K \\ 0, & 0 < K. \end{cases} \]

Hence \( G'(fe^{-\frac{1}{2}\sigma^2 r + \sigma \sqrt{\tau} z}) = 1 \) if \( z > -d_- \), and 0 otherwise, and

\[ \Delta_{\text{Call}}(f, t) = e^{-r\tau} \int_{-d_-}^{\infty} 1 \cdot e^{-\frac{1}{2}\sigma^2 r + \sigma \sqrt{\tau} z} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \]

\[ = e^{-r\tau} \int_{-d_-}^{\infty} e^{-\frac{1}{2}(z - \sigma \sqrt{\tau})^2} \frac{dz}{\sqrt{2\pi}} \]

\[ = e^{-r\tau} \int_{-\infty}^{d_+ + \sigma \sqrt{\tau}} e^{-\frac{1}{2}w^2} \frac{dw}{\sqrt{2\pi}} \]

\[ = e^{-r\tau} \Phi(d_+). \]
Question 2. A risky asset with price $S_t$ pays out a continuous dividend at a constant rate of $q$. We assume that the price-dynamics of the asset is given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\sigma$ and $\mu$ are constants. Let $r$ be the (constant) risk-free rate. Let $P(S_t)$ be the price of an perpetual American put on the asset, as function of the price of the underlying. Assume the strike of the put is equal to $K$.

(a) Show that prior to exercise, $P(S)$ has to satisfy the Ordinary Differential Equation (ODE)

$$\frac{1}{2} \sigma^2 S^2 \frac{d^2 P(S)}{dS^2} + (r - q) S \frac{dP(S)}{dS} = r P(S),$$

and write down the general solution of this ODE. [5 points]

(b) Derive the price of the perpetual American put. [4 points]

(c) Suppose you would have to price an American put on a very long-dated future with maturity $T \gg 1$, where the maturity of the put coincides with that of the future. How might you use the preceding result to obtain an approximate answer? [3 points]

(d) Let $S_\ast = S_\ast(q)$ be the optimal exercise price of the American put.

(i) Examine the behaviour of $S_\ast(q)$ for increasing $q$. 

*Hint:* compute the derivative with respect to $q$. [3 points]

(ii) Give a financial interpretation. [2 points]

(iii) Discuss the behaviour of the perpetual put’s price as the dividend rate increases. [3 points]
Solution:

(a) The standard Black and Scholes hedging argument is applicable as long as the derivative is still traded, i.e. before exercise. Students can either derive the full time-dependent B&S PDE:

\[ \frac{\partial V}{\partial t} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} = rV, \]

and note that the price of a perpetual American option will be independent of \( t \), or directly apply the hedging argument to a derivative whose price is a function of the underlying only.

If one takes a short position of \( \Delta_t \) in a dividend paying asset, the change in value of this position will be

\[ \Delta_t (dS_t + qS_t dt), \]

due to the dividend payed out over \([t, t+dt]\). This accounts for the coefficient of \( \frac{\partial}{\partial S} V \) being \( (r - q) \) instead of \( r \).

The ODE is Euler-type, with general solution \( C_- S^\lambda_- + C_+ S^\lambda_+ \), where \( \lambda_\pm \) are the two roots (in increasing order) of the quadratic equation

\[ \frac{1}{2} \sigma^2 \lambda^2 + \left( r - q - \frac{1}{2} \sigma^2 \right) \lambda - r = 0, \]

or

\[ \lambda_\pm = -\frac{(r - q - \frac{1}{2} \sigma^2) \pm \sqrt{(r - q - \frac{1}{2} \sigma^2)^2 + 2r\sigma^2}}{\sigma^2}. \]

Observe that \( \lambda_- < 0 < \lambda_+ \).

(b) The put will be exercised if \( S \) is below a certain optimal exercise boundary \( S^* \), which we have to determine as part of the solution. The boundary conditions are \( P(S^*) = K - S^* \) and \( \frac{dP}{dS}(S^*) = -1 \) (smooth pasting). For \( S > S^* \), the put price satisfies the ODE of part (a). Since \( P(S) \to 0 \) as \( S \to \infty \), \( C_+ = 0 \), so

\[ P(S) = C_- S^\lambda_, \quad S > S^*. \]

Next,

\[ P(S^*) = K - S^* \Rightarrow C_- = (K - S^*)(S^*)^{-\lambda_-}, \]

and

\[ \frac{dP}{dS}(S^*) = -1 \Rightarrow \lambda_- C_- (S^*)^{-\lambda_- - 1} = -1. \]

Hence

\[ \lambda_- \frac{K - S^*}{S^*} = -1, \]

or

\[ S^* = \frac{\lambda_-}{\lambda_- - 1} K. \]
The price of the put then equals

\[ P(S) = (K - S_*) \left( \frac{S}{S_*} \right)^{\lambda_-}. \]

(c) Mathematically, a derivative on a future is the same as a derivative on an asset paying a continuous dividend with \( q = r \) (and also financially: such an asset is equivalent to a future, since buying such an asset can be completely financed by borrowing money at the risk-free rate: the dividends will cover the interest rate payments). Also, if the maturity \( T \gg t \), we can pretend the option is perpetual, so we simply take the formulas of (b) with \( q = r \) and, consequently,

\[ \lambda_- = \frac{\frac{1}{4}\sigma^2}{\sigma^2} \pm \sqrt{\frac{\frac{1}{8}\sigma^4 + 2r\sigma^2}{\sigma^2}} = \frac{1}{2} \left( 1 - \sqrt{1 + 8r/\sigma^2} \right). \]

(d) (i) The derivative of \( \lambda_-(q) \) with respect to \( q \) is:

\[ \lambda'_-(q) = \sigma^{-2} \left( 1 + \frac{r - q - \frac{1}{2}\sigma^2}{\sqrt{(r - q - \frac{1}{2}\sigma^2)^2 + 2\sigma^2r}} \right) > 0, \]

since the second term in the brackets has absolute value < 1. Next,

\[ S'_*(q) = \frac{-\lambda'_-(q)}{(1 - \lambda_-)^2} K < 0. \]

so the exercise boundary decreases with \( q \).

(ii) For example, if an investor holds the asset but has bought an American put as portfolio insurance, he would postpone exercising the put and selling the asset if the dividend increase, since she/he/it wants to benefit from the dividend payments.

(iii) We have

\[ \log P = \log(K - S_*) + \lambda_- \log(S/S_*) \]

By (i), \( \log(K - S_*) \) increases as function of \( q \). Taking derivatives with respect to \( q \) of the second term:

\[ \lambda'_-(q) \log(S/S_*) - \lambda_-(q) \frac{S'_*(q)}{S_*(q)} > 0, \]

since \( S \geq S_* \) before exercise, and \( \lambda_- < 0 \). The price of the put increases with \( q \).
**Question 3.** Consider a financial market consisting of a single risky asset $S_t$ and a risk-free asset $B_t$, with price-dynamics are given by

$$
\begin{align*}
    dS_t &= \mu S_t dt + \sigma S_t dW_t \\
    dB_t &= r B_t dt, 
\end{align*}
$$

with respect to the objective probability measure $\mathbb{P}$. Here $W_t$ is a Brownian motion and $r$, $\mu$ and $\sigma$ are constants.

(a) Explain how one can use Girsanov’s theorem to show that there exists an equivalent probability measure $\mathbb{Q}$ with respect to which discounted asset prices $\tilde{S}_t = e^{-rt} S_t$ are martingales, with dynamics

$$
    d\tilde{S}_t = \sigma \tilde{S}_t d\tilde{W}_t,
$$

where $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion. [4 points]

(b) Consider a replicable European claim with pay-off $X$ at maturity $T$, and let $(\phi_t, \psi_t)_{0 \leq t \leq T}$ be the self-financing replicating portfolio strategy. Let $\pi_t(X)$ be the price of the claim at $t \leq T$, and let $\tilde{\pi}(X) = B_t^{-1} \pi_t(X)$ be its discounted price.

(i) Briefly sketch why $\tilde{X} = \tilde{\pi}(X) + \int_t^T \phi_u d\tilde{S}_u$. [3 points]

(ii) From (i), deduce the risk-neutral pricing formula for $\pi_t(X)$. [3 points]

(c) Use the risk-neutral pricing formula to derive the put-call parity relationship for European calls and puts. [4 points]

(d) A European chooser option with exercise time $T_c < T$ is the option to choose, at time $T_c$, between a put and a call with identical maturity $T > T_c$ and strike $K$. Its pay-off at $T_c$ is therefore

$$
    \text{Chooser}_{T_c} = \max(P_{T_c}, C_{T_c}),
$$

where $P_t$ and $C_t$ denote the put and call price, respectively, at $t < T$.

(i) By using put-call parity, show that $\text{Chooser}_{T_c} = P_{T_c} + \max \left( S_{T_c} - e^{-r(T_c)} K, 0 \right)$. [3 points]

(ii) Express the price of the chooser option at $t < T_c$ in terms of the price of the put $P_t$ and the price of a suitably defined call option with maturity $T_c$. [3 points]
Solution:

(a) Girsanov’s theorem states that if \( \hat{W}_t = \gamma t + W_t \), then there exists an equivalent probability measure \( Q = Q_\gamma \) with respect to which \( \hat{W}_t \) is again a Brownian motion. In terms of \( \hat{W}_t \),

\[
dS_t = (\mu - \gamma \sigma) S_t dt + \sigma S_t d\hat{W}_t.
\]

If we take \( \gamma = (\mu - r)/\sigma \), then

\[
dS_t = r S_t dt + \sigma S_t d\hat{W}_t,
\]

which implies

\[
d\tilde{S}_t = \sigma \tilde{S}_t d\hat{W}_t.
\]

(b) (i) Let \( V_t = \varphi_t S_t + \psi_t B_t \). Self-financing means:

\[
dV_t = \varphi_t dS_t + \psi_t dB_t,
\]

and the portfolio strategy replicates \( X \) if \( V_T = X \). By absence of arbitrage, we then have \( V_t = \pi_t(X) \): we can reach an amount of \( X \) at time \( T \) by investing \( V_t \) into the self-financing portfolio. The self-financing condition remains valid after discounting (no proof required), so

\[
d\tilde{\pi}_t(X) = d\tilde{V}_t = \varphi_t d\tilde{S}_t
\]

\((d\tilde{B}_t = d(1) = 0)\). The stated formula follows by integration from \( t \) to \( T \).

(ii) Taking the conditional expectation at \( t \) (relative to the Brownian filtration) of both sides of (5), and using that the conditional expectation of the stochastic integral is 0, one finds:

\[
e^{-rt} \pi_t(X) = \mathbb{E}_{Q,t} \left( e^{-rT} X | \mathcal{F}_t \right),
\]

or \( \pi_t(X) = e^{-r(T-t)} \mathbb{E}_{Q,t}(X) \).

(c) Let \( C_t = C_t(K,T) \) be the price at time \( t \leq T \) of a European call with strike \( K \) and maturity \( T \), and \( P_t = P_t(K,T) \) that of the put with same strike and maturity. Then

\[
C_T - P_T = \max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K.
\]

Hence by the risk-neutral pricing formula

\[
C_t - P_t = e^{-r(T-t)} \mathbb{E}_{Q,t} (C_T - P_T) = e^{-r(T-t)} \mathbb{E}_{Q,t} (S_T) - Ke^{-r(T-t)} = S_t - Ke^{-r(T-t)},
\]

since \( \mathbb{E}_{Q,t} (e^{-rT} S_T) = e^{-rt} S_t \), by (a).
(d) (i) Since \( C_{T_c} = P_{T_c} + S_{T_c} - Ke^{-r(T-T_c)} \), the chooser pay-off is equal to
\[
\max \left( P_{T_c}, P_{T_c} + S_{T_c} - Ke^{-r(T-T_c)} \right) = P_{T_c} + \max \left( S_{T_c} - Ke^{-r(T-T_c)}, 0 \right).
\]

(ii) By risk-neutral pricing and part (i), the price of the chooser option at \( t < T_c \) equals
\[
e^{-r(T_c-t)}\mathbb{E}_{Q,t} \left( P_{T_c} \right) + e^{-r(T_c-t)}\mathbb{E}_{Q,t} \left( \max \left( S_{T_c} - Ke^{-r(T-T_c)}, 0 \right) \right)
= P_t + C_t \left( Ke^{-r(T-T_c)}, T_c \right),
\]
the second term on the right being the price at \( t \) of a European call with maturity \( T_c \) and strike \( Ke^{-r(T-T_c)} \).
Question 4. In Merton’s jump diffusion model, the price \( S_t \) of a risky asset evolves according to
\[
dS_t = \mu S_t dt + \sigma S_t dW_t + (J_t - 1)S_t dN_t,
\]
where \( W_t \) is a Brownian motion, \( N_t \) is a Poisson process with intensity \( \lambda \), and \( J_t \) is the jump-size if a jump takes place at time \( t \). It is assumed that all these processes, as well as the jump-sizes at different points in time, are independent. We assume that the jumps are stochastic, and let
\[
\kappa := \lambda \mathbb{E}(J - 1).
\]

(a) Compute the expected rate of return of \( S_t \). [2 points]

(b) By integrating the SDE, show that
\[
S_t = S_0 \left( \prod_{i=0}^{N_t} J_i \right) e^{(\mu - \frac{1}{2} \sigma^2) t + \sigma W_t},
\]
where \( N_t \) is the number of jumps between 0 and \( t \), \( t_i \) are the (random) jump times, in increasing order, and and \( J_i \) is the jump-size at successive jump-times \( t_i \). [4 points]

(c) Suppose you have sold a European derivative on the asset, maturing at \( T \). Explain how to hedge your position. Is it possible to make the hedging portfolio completely risk-free? What part of the risk can be completely hedged, and what part cannot? [5 points]

(d) Under Merton’s assumptions regarding investor’s attitude to jump risk, deduce Merton’s pricing equation for the price \( V(S_t, t) \) of a derivative paying off \( F(S_T) \) at \( T \), where \( F(S) \) is a given function. [4 points]

(e) The Feynman-Kac formula for jump-diffusions implies that
\[
V(S, t) = e^{r T} \mathbb{E} \left( F(\hat{S}_T) \mid \hat{S}_t = S \right),
\]
where \( \hat{S}_t \) is a jump-diffusion (8) with a new parameter \( \mu \) such that mean rate of return becomes equal to \( r \). Show that
\[
V(S, t) = \sum_{k=0}^{\infty} \frac{(\lambda \tau)^k}{k!} e^{-(r+\lambda)\tau} \mathbb{E} \left( F(S J_1 \cdots J_k e^{(r-\kappa-\frac{1}{2} \sigma^2)\tau + \sigma(W_T - W_t)}) \right),
\]
with independent jumps \( J_1, J_2, \ldots \).
How would you use this formula if the jump-intensity is small? [5 points]
Solution:

(a) \( E_t(dN_t) = \mathbb{E}(dN_t) = \lambda dt \), and \( E_t(dW_t) = \mathbb{E}(dW_t) = 0 \), so

\[
E_t(dS_t) = \mu S_t dt + \mathbb{E}(J_t - 1)\lambda S_t dt.
\]

The expected return over \((t, t + dt] \) is therefore

\[
E_t(S_t)/S_t = (\mu + \kappa) dt.
\]

(b) If a jump takes place at \( t_i \), then \( dN_{t_i} = 1 \), and \( S_{t_i} - S_{t_{i-}} = dS_{t_i} = (J - 1)S_{t_{i-}} \), whence \( S_{t_i} = JS_{t_{i-}} \).

Between two jumps \( t_i \leq t < t_{i+1} \) jumps \( S_t \) evolves according to \( \mu S_t dt + \sigma S_t dW_t \) with solution \( S_t = S_{t_i} \exp((\mu - \frac{1}{2} \sigma^2)(t - t_j)) \) while at \( t_i \), \( S_t \) gets multiplied by \( J_{t_i} =: J_i \).

(c) We can only hedge by trading in the underlying asset \( S_t \). Suppose, after having sold the derivative, we go long \( \Delta_t \) of the underlying at \( t \), and let \( V(S_t, t) \) be the value of the derivative, as function of the price of the underlying and time. Then the change in value of our hedged portfolio between \( t \) and \( t + dt \) is

\[
= -dV(S_t, t) + \Delta_t dS_t.
\]

By Ito’s lemma for jump diffusions,

\[
dV(S_t, t) = \partial_t V dt + \partial_S V \, dS_t + \frac{1}{2} \partial^2_S V (dS_t)^2 + (V(S_{t-} + (J_t - 1)S_{t-}) - V(S_{t-})) \, dN_t
\]

\[
= \left( \partial_t V + \mu S_{t-} \partial_S V + \frac{1}{2} \sigma^2 S^2_{t-} \partial^2_S V \right) dt + \sigma S_{t-} \partial_S V dW_t + (V(J_tS_{t-}) - V(S_{t-})) \, dN_t,
\]

where all derivatives are evaluated in \((S_{t-}, t)\), that is, before any jumps.

The risky part in the change in portfolio value is therefore

\[
\sigma (\Delta_t - \partial_S V) S_{t-} dW_t + (\Delta_tS_{t-}(J_t - 1) - (V(J_tS_{t-}) - V(S_{t-}))) \, dN_t.
\]

The jump part cannot be hedged, not even by itself, since the size of the jump is stochastic, and not known before the jump: we would want

\[
V(JS_{t-}, t) - V(S_t, t) - \Delta_t(J - 1)S_{t-} = 0,
\]

but this would require, for the choice of \( \Delta_t \), that we know how big the jump is.

We can only completely hedge away the diffusion risk, by taking \( \Delta_t = \partial_S V(S_{t-}, t) \), the usual \( \Delta \)-hedge.

(d) Merton’s assumption is that investor’s do not require to be regrded for the jump risk, since this is idiosyncratic to the stock and can be diversified
away. The mean rate of return on a ∆-hedged portfolio should then be the risk-free rate of return, \( r \):

\[
\mathbb{E}(-dV(S_t, t) + \partial S V(S_t, t)) = r(−V(S_t, t) + \partial S V(S_t, t))dt.
\]

Working out the expectation on the right then yields Merton’s pricing equation:

\[
\partial_t V + \frac{1}{2} \sigma^2 S^2 \partial^2 S V + (r - \kappa) \partial S V + \lambda \mathbb{E}(V(JS, t) - V(S, t)) = rV,
\]

which should be satisfied prior to exercise: \( t > T \), while at \( T \), \( V(S, T) = F(S) \), the pay-off.

(e) By (a), the \( \mu \) in the SDE for \( \tilde{S}_t \) should be taken equal to \( r - \kappa \). By part (b) and (10),

\[
V(S, t) = e^{-rt} \mathbb{E}_Q \left( F \left( S \left( \prod_{i=0}^{N_T} J_i \right) e^{(r-\kappa \frac{1}{2} \sigma^2)(T-t)+\sigma(W_T-W_t)} \right) \right),
\]

where we used that \( N_T - N_i \) is equal to \( N_{T-t} \) in distribution. The stated formula follows by conditioning on the number of jumps \( k \), and using the definition of the Poisson distribution.

If \( \lambda \) is small, one might just take the first two or three terms of the series.
 SECTION B: Answer TWO questions from this section.)

Question 5.
The different sections of the question below are independent.

Part One.

Consider a digital option paying one million dollar at date \( T = \text{Dec 23, 2014} \) if at that date the share price of Google is higher than \$850, and paying nothing otherwise. You will assume the Black- Scholes setting is valid, including the non-payment of dividends up to date \( T \).

1. Write and explain in all details the price \( C(0) \) of the digital option at the initial date of the contract. \([4 \text{ points}]\)

2. Propose a hedging strategy for the seller of the option at a date \( t \) prior to November 1, 2014. \([2 \text{ points}]\)

3. A call-spread is defined as the combination of a long plain-vanilla call with strike \( L \) and a short plain-vanilla call with strike \( M \) strictly greater than \( L \), same underlying \( S \) and same maturity \( T \).

   (i) Represent the payoff profile at maturity \( T \) of such a call spread. \([2 \text{ points}]\)

   (ii) We consider the hedge of the digital call at a date \( t \) belonging to the week Dec 16 to Dec 23, 2014. Show that for some values of Google at that time, it will be useful to introduce a call-spread in the hedging strategy. \([2 \text{ points}]\)

Part Two.

1. Write and prove the spot forward relationship for an equity index paying a continuous and constant dividend rate \( g \). You will answer successively in the case of constant and stochastic interest rates. \([3 \text{ points}]\)

2. (i) Represent and discuss your P&L (Profit and Loss) at date \( T \) attached to a short forward written on a non-dividend paying stock \( S \), with maturity \( T \); you sold the forward contract at date 0. \([2 \text{ points}]\)

   (ii) List the cases where Forward prices and Futures prices with same underlying and maturity, are equal. \([3 \text{ points}]\)

   (iii) Explain why the Department of a Bank where Forwards and Futures are traded is called Delta One. \([2 \text{ points}]\)
Solution: see hand-written notes, attached.
Question 6. The two sections of this question are independent. You should answer both of them.

Part A.

1. Write and prove \textbf{without} the use of a PDE (partial differential equation) the Black-Scholes formula for calls written on non-dividend paying stocks. [3 points]

2. Write and sketch the proof of the extension of this formula to stochastic interest rates. [2 points]

3. Define and explain the different types of smile and skew which have been observed in the stock market since 1973 up to now. List and explain the different models that have been proposed up to now. [5 points]

Section B:

Let $X_t$ be the Euro / $ exchange rate, defined as the number of dollars necessary to buy one Euro at date $t$. We assume that the process $(X_t)_{t \geq 0}$ is driven under the real probability measure $\mathbb{P}$ by the stochastic differential equation

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t,$$

where $\mu$ and $\sigma$ are strictly positive constants, and where $(W_t)_{t \geq 0}$ is a $\mathbb{P}$-Brownian motion.

1. Given $X_0$, give the expression of $X_t$ for all $t \geq 0$. [2 points]

2. Show that

$$\mathbb{E}(X_t) \geq X_0,$$

where the expectation is computed under $\mathbb{P}$. Give full details. [2 points]

3. Let $Y_t := 1/X_t$. What does $Y_t$ represent, financially? [2 points]

4. Derive the stochastic differential equation satisfied by $Y_t$. [2 points]

5. Show that

$$\mathbb{E}(X_t) \mathbb{E}(Y_t) = e^{\sigma^2 t} \geq 1,$$

for all $t \geq 0$. [2 points]
Solution:

Part A: see handwritten solutions.

Part B:

1. \( X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2) t + \sigma W_t} \).
2. \( e^{-\frac{1}{2}\sigma^2 t + \sigma W_t} \) is a martingale, so \( \mathbb{E}(X_t) = e^{\mu t} X_0 \).
3. \( Y_t \) is the $/Euro exchange rate.
4. \( Y_t = e^{(-\mu + \frac{1}{2}\sigma^2) t - \sigma W_t} \), so
   \[ dY_t = (-\mu + \sigma^2) Y_t dt - \sigma Y_t dW_t. \]
5. This follows from \( \mathbb{E}(Y_t) = e^{(-\mu + \sigma^2) t} Y_0 \) and \( Y_0 = X_0^{-1} \).
7. Consider a generic short rate model, with risk-neutral evolution of the short rate \( r_t \) given by
\[
dr_t = a(r_t)dt + b(r_t)d\hat{W}_t.
\]
(11)

(a) (i) Briefly explain why the price at time \( t \) of a 0-coupon bond maturing at \( T \) is given by
\[
p(t, r_t) = \mathbb{E}_{Q_t} \left( e^{-\int_t^T r_s ds} | r_t \right).
\]
(2 points)

(ii) What PDE does the function \( p(r, t) \) have to satisfy for \( t < T \), and what is the boundary condition at \( T \)? (3 points)

(b) We now specialise to the Vasicek model:
\[
dr_t = \alpha(\theta - r_t)dt + \sigma d\hat{W}_t.
\]
(12)

It is known that the Vasicek model belongs to the class of affine models, for which 0-coupon prices can be expressed as
\[
p(t, r) = e^{A(t,T)-B(t,T)r},
\]
(13)
where \( A(t, T) \) and \( B(t, T) \) are known of time and maturity.

Derive an ODE which \( B(t, T) \) has to satisfy, and from that show that
\[
B(t, T) = \alpha^{-1} \left( 1 - e^{-\alpha(T-t)} \right).
\]
(14) (5 points)

(c) Consider the risk-neutral evolution of the zero-coupon bond prices:
\[
dp(t, T) = \alpha_{t,T} p(t, T)dt + \beta_{t,T} p(t, T)d\hat{W}_t,
\]
with coefficients \( \alpha_{t,T}, \beta_{t,T} \), we want to determine.

(i) Without computations, what is \( \alpha_{t,T} \) is equal to, and why? (2 points)

(ii) Show that for the Vasicek model,
\[
\beta_{t,T} = -\sigma B(t, T).
\]
(3 points)

(d) (i) Give an expression for the \( T \)-forward price of a 0-coupon bond of maturity \( S > T \). (2 points)

(ii) Show that the dynamics of the \( T \)-forward \( F_t \) price with respect to the \( T \)-forward measure in the Vasicek model is given by
\[
dF = \sigma (B(t, T) - B(t, S)) F_idW^T_i,
\]
where \( W^T_i \) is a Brownian motion with respect to \( F^T \). (3 points)
Solution:

(a) (i) Risk-neutral pricing.
(ii) By Feynman - Kac, \( p(r, t) \) has to satisfy
\[
\partial_t p + \frac{1}{2} b(r)^2 \partial_r^2 p + a(r) \partial_r p = rp, \quad t < T,
\]
with boundary condition \( p(r, T) = 1 \).

(b) Substituting \( p = e^{A - Br} \) into the PDE gives (with the prime denoting the derivative with respect to \( t \)),
\[
(A' - rB') + \frac{1}{2} \sigma^2 B^2 - \alpha(\theta - r)B = r.
\]
Separating out the terms which only depend on \( t \) and those of the form \( r \cdot f(t) \) and noting these both have to be identically 0 (or, somewhat quicker, differentiating the above expression with respect to \( r \)) we find that
\[
B' = \alpha B - 1, \quad t < T.
\]
Since \( B = 0 \) when \( t = T \), the solution is as stated (by inspection).

(c) (i) With respect to the risk-neutral measure, all traded assets have the risk-free rate as return, so \( \alpha_{t,T} = r_t \).
(ii) By Ito’s lemma,
\[
dp\left(r_t, t\right) = \left(\cdots\right) dt + \partial_r \left(r_t, t\right) dr_t
\]
\[
= \left(\cdots\right) dt - \sigma B(t, T)e^{A(t,T)-B(t,T)r_t}d\hat{W}_t
\]
\[
= \left(\cdots\right) dt - \sigma B(t, T)p(r_t, t)d\hat{W}_t,
\]
so \( \beta_{t,T} = -\sigma B(t, T) \).

(d) (i) \( T_t = F_t(T; S) = P_{t,S}/P_{t,T} \).
(ii) One computes that, with respect to the risk-neutral measure,
\[
d\left(\frac{P_{t,S}}{P_{t,T}}\right) = \frac{dP_{t,S}}{P_{t,T}} + P_{t,S}d\left(\frac{1}{P_{t,T}}\right) + (dP_{t,S})d\left(\frac{1}{P_{t,T}}\right)
\]
\[
= \frac{dP_{t,S}}{P_{t,T}} - P_{t,S} \frac{dP_{t,T}}{P_{t,T}^2} + (\cdots)dt
\]
\[
= (\beta_{t,S} - \beta_{t,T}) \frac{P_{t,S}}{P_{t,T}} d\hat{W}_t + (\cdots)dt.
\]
Since \( P_{t,S}/P_{t,T} \) is a martingale with respect to \( \mathbb{F}^T \), and the drift can be made to disappear by a Girsanov transformation, we find that
\[
d\left(\frac{P_{t,S}}{P_{t,T}}\right) = \sigma (B(t, T) - B(t, S)) \frac{P_{t,S}}{P_{t,T}} dW_t^T,
\]
where we used part (ii) of (c).
Question 8. In this question, $P_{t,T}$ denotes the price of a 0-coupon bond maturing at $T$. Let $T_0 < T_1 < \ldots < T_n$ be a set of bond maturities (tenor dates). Recall that if $t \leq T_{i-1}$, then the LIBOR forward rate $L_t(T_{i-1}, T_i)$ is defined as the simply compounded interest rate, set at $t \leq T_{i-1}$ for borrowing/lending over the future period $[T_{i-1}, T_i]$.

(a) Show using an absence of arbitrage argument that

$$L_t(T_{i-1}, T_i) = \frac{1}{\delta_i} \left( \frac{P_{t,T_i}}{P_{t,T_{i-1}}} - 1 \right), \quad \delta_i = T_i - T_{i-1}. \tag{15}$$

[4 points]

(b) State the defining characteristic of the Forward Neutral Measure $\mathbb{F}^{T_i}$ at $T_i$, and show that

$$L_t(T_{i-1}, T_i) = \mathbb{E}_{t}^{T_i} (L(T_{i-1}, T_i))$$

where $\mathbb{E}_{t}^{T_i}$ is the expectation with respect to $\mathbb{F}^{T_i}$, and where $L(T_{i-1}, T_i) := L_{T_{i-1}}(T_{i-1}, T_i)$ is the LIBOR-rate set at $T_{i-1}$. The sub-index $t$ denotes conditioning with respect to price-information available at $t$, as usual.

[4 points]

(c) Determine the price at $t \leq T_0$ of a Floating Rate Note which pays out a coupon of $\delta_i L(T_{i-1}, T_i)$ at dates $T_1, T_2, \ldots, T_n$ as well as the principal of £1 at $T_n$.

[4 points]

(d) Define and price a fixed-rate payer swap with payment dates $T_1, \ldots, T_n$ and reset dates $T_0, \ldots, T_n$. How should the swap rate be set at the start of the contract, $T_0$? And prior to that, at $t < T_0$?

[4 points]

(e) Give the definition of an interest rate cap and interest rate floor with reset and payment dates $T_0, \ldots, T_{n-1}$ and $T_1, \ldots, T_n$ and strike $K$, and explain how these can be used to manage interest rate risk. Derive a parity relationship between caps, floors and swaps.

[4 points]
Solution:

(a) LIBOR forward rate: rate set at \( t \leq T_{i-1} \) for borrowing over \([T_{i-1}, T_i]\). Consider two different ways of obtaining £1 at \( T_i \):

1. Selling a 0-coupon bond with maturity \( T_i \);
2. Selling \((1 + \delta_j L_t(T_{i-1}, T_i))^{-1}\) 0-coupons with maturity \( T_{i-1} \) and entering into a forward rate agreement over \([T_{i-1}, T_i]\) at \( L_t(T_{i-1}, T_i) \).

Both strategies should cost the same, so \( P_{t,T_i} = (1 + \delta_j L_t(T_{i-1}, T_i))^{-1} P_{t,T_{i-1}} \), etc.

(b) If \( X_t \) is the price of any traded asset, then \( X_t / P_{t,T_i} \) is a martingale with respect to \( F_{T_i} \). In particular, \( L_t(T_{i-1}, T_i) \) is a martingale (interpreted to be constant over \([T_{i-1}, T_i]\)), which implies the stated formula.

(c) Two solutions are possible:

**Solution 1.** Using the forward neutral pricing formula, the value at \( t \leq T_0 \) of the cash-flow \( \delta_i L(T_{i-1}, T_i) \) at \( T_i \) is

\[
\delta_i P_{t,T_i} \mathbb{E}_t^{T_i} \left( L(T_{i-1}, T_i) \right) = \delta_i P_{t,T_i} L_t(T_{i-1}, T_i) = P_{t,T_{i-1}} - P_{t,T_i},
\]

while the final cash-flow of 1 at \( T_n \) is worth \( P_{t,T_n} \) at \( t \). hence the value of the FRN at \( t \) is

\[
\sum_{i=1}^{n} (P_{t,T_{i-1}} - P_{t,T_i}) + P_{t,T_n} = P_{t,T_0}.
\]

**Solution 2.** The floating rate note can be hedged using the following strategy

- At \( T_0 \), take the £1 and invest it in bonds of maturity \( T_1 \): this will earn an interest of

\[
\frac{1}{\delta_1} \left( \frac{1 - P_{T_0}}{P_{T_0}} \right) = L(T_0, T_1)
\]

- At \( T_1 \), collect \((1 + \delta_1 L(T_0, T_1))\), pay out the required coupon of \( \delta_1 L(T_0, T_1) \), and invest the remaining 1 Sterling in maturity \( T_2 \)-bonds, whose yield over \([T_1, T_2]\) is exactly \( L(T_1, T_2) \)

- At \( T_2 \), collect \((1 + \delta_2 L(T_1, T_2))\), pay out the coupon, and invest the remaining 1 Sterling in \( T_3 \)-bonds.

- Repeat this until time \( T_N \), at which time you pay out the final coupon, together with the 1 Sterling you had to deliver.
Hence the floating bond’s price at \( T_0 \) is exactly the cost of this strategy, which is £1 (if not, set up an arbitrage by shorting the more expensive of the two, and going long the cheaper).

If \( t < T_0 \), buy a \( T_0 \)-bond to get 1 at \( T_0 \) ⇒ floating rate’s price at \( t \) is \( P_{t,T_0} \).

(d) A fixed-for-floating swap exchanges interest rate payments of a bond with a fixed coupon rate of \( k \) and a FRN with the same principal. If we normalise the principals to be 1, the value of the swap at \( t \leq T_0 \) would exactly be the difference between the two bond-prices:

\[
\text{FRN}_t - \sum_{i=1}^{n} \delta_i k P_{t,T_i} + P_{t,T_n}.
\]

By part (c), the right hand side is exactly \( P_{t,T_0} \). Since the first cash exchange takes place at \( T_1 \), the price of the swap at \( T_0 \) should be 0, which leads to a fair swap rate of

\[
k = s_{T_0} := \frac{1 - \sum_{i=1}^{n} P_{T_0,T_i}}{\sum_{i=1}^{n} P_{T_0,T_i}}.
\]

Similarly, for \( t < T_0 \) we find the forward swap rate

\[
s_t := \frac{P_{t,T_0} - \sum_{i=1}^{n} P_{t,T_i}}{\sum_{i=1}^{n} P_{T_0,T_i}}.
\]

(e) A cap with strike \( K \) pays out \( \delta_i \max(L(T_{i-1}, T_i) - K, 0) \) at \( T_i, i = 1, \ldots, n \), and a floor pays out \( \delta_i \max(K - L(T_{i-1}, T_i)) \). A cap can be used to effectively bound your interest by \( K \), if you happen to have to pay interest on a FRN, since it will pay you back all the interest you payed above \( K \). Similarly, a floor will guarantee you a minimum interest of \( K \), if you receive interests on FRN.

Since

\[
\max(L(T_{i-1}, T_i) - K, 0) - \max(K - L(T_{i-1}, T_i)) = L(T_{i-1}, T_i) - K,
\]

being long a cap and short a floor is the same as holding a fixed-for-floating swap with fixed rate \( K \). Hence their prices satisfy

\[
\pi_t^{\text{Cap}} - \pi_t^{\text{Floor}} = \pi_t^{\text{Swap}}(K),
\]

the latter having been determined in part (c).