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By

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DELTA-MATROIDS AS SUBSYSTEMS OF SEQUENCES OF HIGGS LIFTS

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ABSTRACT. In [30], Tardos studied special delta-matroids obtained from sequences of Higgs lifts; these are the full Higgs lift delta-matroids that we treat and around which all of our results revolve. We give an excluded-minor characterization of the class of full Higgs lift delta-matroids within the class of all delta-matroids, and we give similar characterizations of two other minor-closed classes of delta-matroids that we define using Higgs lifts. We introduce a minor-closed, dual-closed class of Higgs lift delta-matroids that arise from lattice paths. It follows from results of Bouchet that all delta-matroids can be obtained from full Higgs lift delta-matroids by removing certain feasible sets; to address which feasible sets can be removed, we give an excluded-minor characterization of delta-matroids within the more general structure of set systems. Many of these excluded minors occur again when we characterize the delta-matroids in which the collection of feasible sets is the union of the collections of bases of matroids of different ranks, and yet again when we require those matroids to have special properties, such as being paving.

1. INTRODUCTION

A *set system* is a pair $S = (E, \mathcal{F})$, where E, or $E(S)$, is a set, called the *ground set*, and $\mathcal{F}$, or $\mathcal{F}(S)$, is a collection of subsets of E. (All set systems in this paper have finite ground sets.) The members of $\mathcal{F}$ are the *feasible sets*. We say that S is *proper* if $\mathcal{F} \neq \emptyset$, and that S is *even* if $|X| - |Y|$ is even for all $X, Y \in \mathcal{F}$. A matroid M has many associated set systems with $E = E(M)$ since we can take $\mathcal{F}$ to be, for example, the set $\mathcal{B}(M)$ of its bases, or the set of its independent sets, or the set of its circuits; the first two are always proper. The first is of most interest here since the definition of a delta-matroid can be motivated by an exchange property that the bases of any matroid M satisfy, namely, for any $B_1, B_2 \in \mathcal{B}(M)$ and for each element $x \in B_1 - B_2$, there is a $y \in B_2 - B_1$ for which $B_1 \triangle \{x, y\} \in \mathcal{B}(M)$. To get the definition of a delta-matroid, replace set differences by symmetric differences. Thus, as introduced by Bouchet in [8], a *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ for which $\mathcal{F}$ satisfies the *delta-matroid symmetric exchange axiom*:

(SE) for all triples (X, Y, u) with X and Y in $\mathcal{F}$ and $u \in X \triangle Y$, there is a $v \in X \triangle Y$ (perhaps u itself) such that $X \triangle \{u, v\}$ is in $\mathcal{F}$.

Just as there is a mutually-enriching interplay between matroid theory and graph theory, the theory of delta-matroids has substantial connections with the theory of embedded graphs; see [14, 15].

Naturally, there are strong links between matroids and delta-matroids; below we cite several that are relevant in this paper. First, for a delta-matroid D, let $\max(\mathcal{F}(D))$ be the collection of sets in $\mathcal{F}(D)$ that have the largest cardinality among sets in $\mathcal{F}(D)$, and define $\min(\mathcal{F}(D))$ similarly. An easy application of property (SE) shows that each of $\max(\mathcal{F}(D))$ and $\min(\mathcal{F}(D))$ is the collection of bases of a matroid on E; we denote these matroids by $D_{\max}$ and $D_{\min}$, respectively, and call them the *maximal* and *minimal matroids* of D.

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A matroid Q on E is a *quotient* of a matroid L on E, or L is a *lift* of Q, if there is a matroid M and a subset X of $E(M)$ for which $M\backslash X = L$ and $M/X = Q$. The following connection between $D_{\min}$ and $D_{\max}$ was proven by Bouchet [11, Theorem 3.3].

Proposition 1.1. *For any delta-matroid D, the matroid $D_{\min}$ is a quotient of $D_{\max}$.*

This result and the following property of $D_{\min}$ and $D_{\max}$ are important in our work.

Lemma 1.2. *If X is any feasible set in a delta matroid D, then there are bases B' of $D_{\min}$ and B of $D_{\max}$ with $B' \subseteq X \subseteq B$.*

Proof. Pick a basis B of $D_{\max}$ with $|X \cap B|$ maximal. If $X \not\subseteq B$, then pick $u \in X - B$. Thus, there is a $v \in X \triangle B$ with $B \triangle \{u, v\} \in \mathcal{F}$. Since $B \in \max(\mathcal{F}(D))$, we must have $v \in B - X$, so $B \triangle \{u, v\}$ is a basis of $D_{\max}$. However $|X \cap (B \triangle \{u, v\})| > |X \cap B|$, contrary to the choice of B. Thus, $X \subseteq B$. The existence of B' follows by a similar argument, or by duality, which we discuss in the next section. $\square$

The converse of Proposition 1.1 is true. One way to show it is to show that if Q is a quotient of L, with both matroids on the set E, and if we let $\mathcal{F}$ be the set of all subsets X of E for which there are bases $B' \in \mathcal{B}(Q)$ and $B \in \mathcal{B}(L)$ with $B' \subseteq X \subseteq B$, then $(E, \mathcal{F})$ is a delta-matroid. Such delta-matroids were studied by Tardos in [30]; she called them generalized matroids. In Section 3, we interpret the construction of these special delta-matroids using the Higgs lifts of Q toward L; thus, we call such delta-matroids full Higgs lift delta-matroids. We consider beginning with a full Higgs lift delta-matroid and removing all of the feasible sets of certain cardinalities. We call this a Higgs lift delta-matroid, or an even Higgs lift delta-matroid when all of the feasible sets of one parity are removed. (See Proposition 3.1.) We give an excluded-minor characterization of Higgs lift delta-matroids (Theorem 3.4), as well as counterparts in the full case and in the even case. In Section 4, we introduce Higgs lift delta-matroids that arise from lattice paths.

Lemma 1.2 says that any delta-matroid can be obtained from a full Higgs lift delta-matroid by discarding some of the feasible sets. It is natural to ask what restrictions there are on the sets that we remove. This issue is addressed in Section 5, where we give an excluded-minor characterization of delta-matroids within the broader structure of set systems. We address the corresponding issues for even delta-matroids, for matroids, and for binary delta-matroids.

For a delta-matroid D and any integer i with $r(D_{\min}) \leq i \leq r(D_{\max})$, let N_i be the set system $(E, \{F \in \mathcal{F} : |F| = i\})$. If D is a Higgs lift delta-matroid, then each proper set system N_i is a matroid, but this need not be true for other delta-matroids. In Section 6, we characterize the delta-matroids D for which each N_i is a matroid, as well as, for instance, when each N_i is a paving matroid or a sparse paving matroid.

We follow the notation and terminology for matroids that is used in [27]. In the next section, we review some key points about delta-matroids, as well as some of the more specialized matroid topics that play roles throughout this paper.

2. Background

2.1. **Minors and twists of set systems.** Let $S = (E, \mathcal{F})$ be a proper set system. An element $e \in E$ is a *loop* of S if no set in $\mathcal{F}$ contains e. If e is in every set in $\mathcal{F}$, then e is a *coloop*. If e is not a loop, then the *contraction of e from S*, written S/e, is given by

$$S/e = (E - e, \{F - e : e \in F \in \mathcal{F}\}).$$

(As in matroid theory, we usually omit set brackets from singleton sets.) If e is not a coloop, then the *deletion of e from S*, written $S\backslash e$, is given by

$$S\backslash e = (E - e, \{F \subseteq E - e : F \in \mathcal{F}\}).$$

If e is a loop or a coloop, then one of S/e and $S\backslash e$ has already been defined, so we can set $S/e = S\backslash e$. Any sequence of deletions and contractions, starting from S, gives another set system S', called a *minor* of S. Each minor of S is a proper set system. Note that if S is even, then so are its minors.

A collection $\mathcal{C}$ of proper set systems is *minor closed* if every minor of every member of $\mathcal{C}$ is in $\mathcal{C}$. Given such a collection $\mathcal{C}$, a proper set system S is an *excluded minor* for $\mathcal{C}$ if $S \notin \mathcal{C}$ and all other minors of S are in $\mathcal{C}$. A proper set system belongs to $\mathcal{C}$ if and only if none of its minors is an excluded minor for $\mathcal{C}$. Thus, the excluded minors determine $\mathcal{C}$; they are the minor-minimal obstructions to membership in $\mathcal{C}$.

The order in which elements are deleted or contracted can matter since, for instance, contracting an element e can turn a non-loop of S into a loop of S/e. For example, if $S = (\{a, b, c, d\}, \{\{a, b\}, \{c, d\}\})$, then c is a loop of S/a and $S/a/c = (\{b, d\}, \{b\})$, whereas a is a loop of S/c and $S/c/a = (\{b, d\}, \{d\})$. However, for disjoint subsets X and Y of E, if some set in $\mathcal{F}$ is disjoint from X and contains Y, then the deletions and contractions in $S\backslash X/Y$ can be done in any order, and

$$S\backslash X/Y = (E - (X \cup Y), \{F - Y \,:\, F \in \mathcal{F} \text{ and } Y \subseteq F \subseteq E - X\}).$$

We next show that all minors of a proper set system are of this type.

Lemma 2.1. *For any minor S' of a proper set system $S = (E, \mathcal{F})$, there are disjoint subsets X and Y of E such that*

$$S' = S\backslash X/Y = (E - (X \cup Y), \{F - Y \,:\, F \in \mathcal{F} \textit{ and } Y \subseteq F \subseteq E - X\}).$$

Proof. Suppose we get S' from S by, for each of $e_1, e_2, \ldots, e_k$ in turn, either deleting or contracting e_i, giving the sequence of minors $S_0 = S, S_1, S_2, \ldots, S_k = S'$. Let X be the set of elements e_i in $\{e_1, e_2, \ldots, e_k\}$ that satisfy at least one of the following conditions:

(1) e_i is a loop of S_{i-1} (so $S_i = S_{i-1}\backslash e_i$), or
(2) e_i is not a coloop of S_{i-1} and $S_i = S_{i-1}\backslash e_i$.

Let $Y = \{e_1, e_2, \ldots, e_k\} - X$, so for each $e_j \in Y$, either e_j is a coloop of S_{j-1} (so $S_j = S_{j-1}/e_j$), or e_j is not a loop of S_{j-1} and $S_j = S_{j-1}/e_j$. Since S' is proper, some set in $\mathcal{F}$ is disjoint from X and contains Y, so the lemma follows from the remarks above. □

Bouchet and Duchamp [12] showed that if S is a delta-matroid and $S' = S\backslash X/Y$, then S' is a delta-matroid and S' is independent of the order of the deletions and contractions.

For $A \subseteq E$, the *twist of S on A*, which is also called the *partial dual of S with respect to A*, denoted $S * A$, is given by

$$S * A = (E, \{F \triangle A \,:\, F \in \mathcal{F}\}).$$

Note that $S/e = (S * e)\backslash e$ and $(S * A) * A = S$. The *dual* S^* of S is $S * E$. Note that twists of even set systems are even. However, apart from the dual, the twists of a matroid $\big(E(M), \mathcal{B}(M)\big)$ are generally not matroids, as discussed in [15, Theorem 3.4].

2.2. **Quotients, lifts, and Higgs lifts.** We will use the following result about quotients, which is well known (see, e.g., [13, 27]).

Lemma 2.2. *For matroids Q and L on E, the statements below are equivalent.*

(1) *The matroid Q is a quotient of L.*
(2) *The matroid L^* is a quotient of Q^*.*
(3) *Each circuit of L is a union of circuits of Q.*
(4) *For each basis B of L and element $e \in E - B$, there is a basis B' of Q with $B' \subseteq B$ and*

$$\{f : (B' \cup e) - f \text{ is a basis of } Q\} \subseteq \{f : (B \cup e) - f \text{ is a basis of } L\}.$$

We will use Higgs lifts, for which we recall only the background we need. (See [7, 13, 23] for more about this construction.) Let Q be a quotient of L on E and set $k = r(L) - r(Q)$. For each integer i with $0 \leq i \leq k$, the function r_i that is defined by

$$r_i(X) = \min\{r_Q(X) + i, r_L(X)\}, \tag{1}$$

for $X \subseteq E$, is the rank function of a matroid on E; this matroid is the *i-th Higgs lift of Q toward L* and is denoted $H^i_{Q,L}$. Its bases are the sets of size $r(Q) + i$ that span Q and are independent in L, or equivalently contain a basis of Q and are themselves contained in a basis of L. It follows that if $0 \leq i \leq j \leq k$, then $H^j_{Q,L}$ is the $(j - i)$-th Higgs lift of $H^i_{Q,L}$ toward L. The matroid $H^i_{Q,L}$ is the freest (i.e., greatest in the weak order) quotient of L that has Q as a quotient and has rank $r(Q) + i$. Higgs lifts commute with minors and duals, as we state next. (See [7, Propositions 2.2 and 2.6] for proofs.) So that we do not need to restrict i and j below, as is common we set $H^i_{Q,L}$ to L when $i > k$, and to Q when $i < 0$.

Lemma 2.3. *If Q is a quotient of L and $i + j = r(L) - r(Q)$, then $(H^i_{Q,L})^* = H^j_{L^*,Q^*}$. Also, if $X \subseteq E$, then $(H^i_{Q,L})|X = H^i_{Q|X,L|X}$ and $(H^i_{Q,L})/X = H^{i-t}_{Q/X,L/X}$ where $t = r_L(X) - r_Q(X)$.*

3. Higgs lift delta-matroids

It is often useful to view a simple graph on n vertices as a subgraph of the maximal such graph, K_n. Similarly, a rank-r simple matroid that is representable over $GF(q)$ can be seen as a restriction of the maximal such matroid, $PG(r-1, q)$. In that spirit, by the next two results we can view each delta-matroid D as coming from the maximal delta-matroid that has the same minimal and maximal matroids as D. These maximal delta-matroids are the case $K = \{0, 1, \ldots, k\}$ in the next result. This result shows that the converse of Proposition 1.1 holds.

Proposition 3.1. *Fix a matroid L on E and a quotient Q of L. Set $k = r(L) - r(Q)$ and let K be a subset of $\{0, 1, 2, \ldots, k\}$ for which $\{0, 1, 2, \ldots, k\} - K$ contains no pair of consecutive integers. Then the union*

$$\mathcal{F} = \bigcup_{i \in K} \mathcal{B}(H^i_{Q,L})$$

of the sets of bases of the Higgs lifts $H^i_{Q,L}$ of Q towards L, indexed by element of K, is the set of feasible sets of a delta-matroid on E.

Proof. With the first part of Lemma 2.3 and the observation that $H^i_{Q,L}, H^{i+1}_{Q,L}, \ldots, H^j_{Q,L}$ are the Higgs lifts of $H^i_{Q,L}$ toward $H^j_{Q,L}$, we may assume that $\{0, k\} \subseteq K$, and it suffices to check property (SE) for all triples (X, Y, u), where $X \in \mathcal{B}(Q)$ and $Y \in \mathcal{B}(L)$ and $u \in X \triangle Y$. Bases of L span Q, so Y spans Q. If $u \in X - Y$, then, since Y spans Q, there is a $v \in Y - X$ for which $(X - u) \cup v$ is a basis of Q, so property (SE) holds. Now assume that $u \in Y - X$. Note that by the hypothesis, K contains either 1 or 2. First assume that $X \cup u$ is independent in L. Thus, $X \cup u$ is a basis of $H^1_{Q,L}$, so taking $v = u$ verifies property (SE) if 1 is in K. Note that $X \cup u$ is independent in $H^2_{Q,L}$ and Y spans $H^2_{Q,L}$, so there is a $v \in Y - (X \cup u)$ with $X \cup \{u, v\} \in \mathcal{B}(H^2_{Q,L})$, so property (SE) holds if 2 is in K. Now assume that $X \cup u$ is dependent in L, so it contains a unique circuit, say C, of L. Since Y is a basis of L, we have $C \not\subseteq Y$, so fix a $v \in C - Y$. By part (3) of Lemma 2.2, C is a union of circuits of Q, and since X is a basis of Q, the set $X \cup u$ contains a unique circuit of Q, so C is a circuit of Q. Now $v \in X - Y$ and $(X \cup u) - v$ is a basis of Q, as needed. □

We call the delta-matroids identified in Proposition 3.1 *Higgs lift delta-matroids*. If $K = \{0, 1, 2, \ldots, k\}$, we have the *full Higgs lift delta-matroid of the pair* (Q, L); they were studied by Tardos [30], who called them generalized matroids, and more recently in [18], where they are called saturated delta-matroids. If k and all elements of K are even, we have the *even Higgs lift delta-matroid of the pair* (Q, L).

It is straightforward to obtain the following characterization of the feasible sets in a Higgs lift delta-matroid.

Lemma 3.2. *A delta-matroid $D = (E, \mathcal{F})$ is a Higgs lift delta-matroid if and only if, for every set $F \subseteq E$, one of the following holds:*

(1) *no set in $\mathcal{F}$ has cardinality $|F|$ or*
(2) *$F \in \mathcal{F}$ exactly when there exist sets $A \in \mathcal{B}(D_{\min})$ and $B \in \mathcal{B}(D_{\max})$ such that $A \subseteq F \subseteq B$.*

The next result follows from Lemma 1.2 and the description of the bases of Higgs lifts.

Corollary 3.3. *If X is a feasible set in a delta-matroid D and $i = |X| - r(D_{\min})$, then X is a basis of the i-th Higgs lift of $D_{\min}$ toward $D_{\max}$. Thus, D is obtained from the full Higgs lift delta-matroid of the pair $(D_{\min}, D_{\max})$ by removing some feasible sets that are not in $\mathcal{B}(D_{\min}) \cup \mathcal{B}(D_{\max})$.*

Theorem 5.1 addresses the question of which feasible sets of the Higgs lift delta-matroid of a pair (Q, L) can be removed to yield delta-matroids.

Now we give an excluded-minor characterization of Higgs lift delta-matroids. We will use the following seven delta-matroids:

- $U_1 = (\{a, b\}, \{\emptyset, \{a\}, \{a, b\}\})$,
- $U_2 = (\{a, b, c\}, \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}\})$,

and, for $3 \leq i \leq 7$, the even delta-matroid U_i has ground set $E = \{a, b, c, d\}$ and its feasible sets are $\emptyset$, E, and the 2-element sets given by the edges of the graph G_i in Figure 1.

Theorem 3.4. *A delta-matroid is a Higgs lift delta-matroid if and only if it has no minor isomorphic to any of $U_1, U_2, \ldots, U_7$.*

The proof of the theorem is postponed until Section 5. This result gives part of the next corollary; the rest is easy to check. The duality assertion uses the first part of Lemma 2.3.

Corollary 3.5. *The classes of Higgs lift delta-matroids, full Higgs lift delta-matroids, and even Higgs lift delta-matroids are closed under minors and duals.*

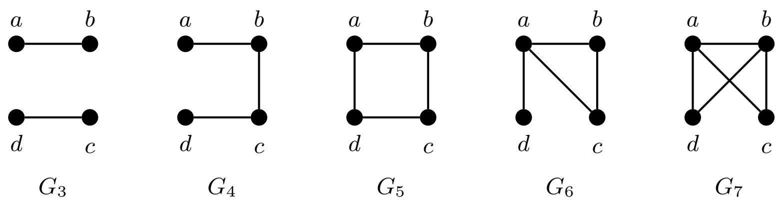


FIGURE 1. The graphs whose edges give the proper, nonempty feasible sets of U_3, U_4, U_5, U_6, and U_7, respectively.

Let S_2 be the delta-matroid $(\{a,b\},\{\emptyset,\{a,b\}\})$. We now characterize full Higgs lift delta-matroids and even Higgs lift delta-matroids by their excluded minors.

Corollary 3.6. *A delta-matroid is a full Higgs lift delta-matroid if and only if it contains no minor isomorphic to U_1 or S_2.*

Proof. It is straightforward to check that U_1 and S_2 are excluded minors for the class of full Higgs lift delta-matroids.

Suppose that the delta-matroid $D=(E,\mathcal{F})$ is not a full Higgs lift delta-matroid. If D is not a Higgs lift delta-matroid, then it has a minor in $\{U_1,U_2,\dots,U_7\}$ and each of $U_2,U_3,\dots,U_7$ has a minor isomorphic to S_2. Suppose that D is a Higgs lift delta-matroid but not a full Higgs lift delta-matroid. For i with $0\le i\le r(D_{\max})-r(D_{\min})$, let N_i be the set system $(E,\{F\in\mathcal{F}:|F|=i+r(D_{\min})\})$. Then for some i with $0<i<r(D_{\max})-r(D_{\min})$, the set system N_i is improper. Both N_{i-1} and N_{i+1} must be proper in order for D to be a delta-matroid. Choose bases B_Q and B_L of $D_{\min}$ and $D_{\max}$ respectively with $B_Q\subseteq B_L$. Then there are sets X and Y belonging to N_{i-1} and N_{i+1} respectively, satisfying $B_Q\subseteq X\subseteq Y\subseteq B_L$. So $D/X\setminus(E-Y)$ is isomorphic to S_2. □

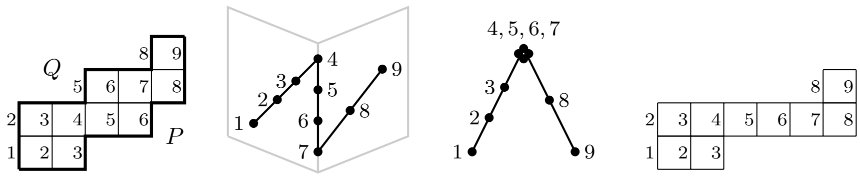
The next corollary follows because a delta-matroid is both even and a Higgs lift delta-matroid if and only if it is an even Higgs lift delta-matroid.

Corollary 3.7. *An even delta-matroid is an even Higgs lift delta-matroid if and only if it contains no minor isomorphic to U_3,U_4,U_5,U_6, or U_7.*

4. LATTICE PATH DELTA-MATROIDS

In this section we define a class of full Higgs lift delta-matroids using lattice paths. This is a natural direction in which to extend the theory of lattice path matroids, which has proven to be a rich vein; for instance, see [1, 3, 5, 16, 17, 19, 21, 22, 25, 26, 28, 29]. The concrete nature of the delta-matroids defined below may help readers get a better handle on delta-matroids, and it may suggest new avenues of investigation.

We first recall lattice path matroids from [6]. (See Figure 2 for illustrations.) The lattice paths that we consider are sequences of steps in $\mathbb{R}^2$, each of unit length, each going north, N, or east, E. Fix two lattice paths P and Q from $(0,0)$ to a point (m,r), where P never rises above Q. Thus, for each i with $1\le i\le r$, if the ith north step of P is in position b_i in P, and the ith north step of Q is in position a_i in Q, then $a_i\le b_i$. The paths P and Q bound a region $\mathcal{R}$ in $\mathbb{R}^2$; let $\mathcal{P}$ be the set of lattice paths from $(0,0)$ to (m,r) that remain in $\mathcal{R}$. For $P'\in\mathcal{P}$, viewed as a word in the alphabet $\{E,N\}$, let $b(P')$ be the set of positions in P' where N occurs. Note that the position, in a lattice path, of any step that ends at (s,t) is $s+t$, so if we put the label $s+t$ on the line segment (a north step) from $(s,t-1)$ to (s,t), then $b(P')$ is the set of labels on the north steps in the path P'. As shown in [6], the



(a) (b) (c) (d)

FIGURE 2. Examples of (a) the region of interest, (b) the lattice path matroid it gives, which is the transversal matroid that has the presentation $\{\{1,2,3\},\{2,\dots,6\},\{5,\dots,8\},\{8,9\}\}$, (c) a quotient of that matroid, and (d) a region that yields the quotient.

set $\{b(P') : P' \in \mathcal{P}\}$ is the set of bases of a transversal matroid, denoted by $M[P,Q]$, and one presentation of this transversal matroid is given by $\{\{a_i, a_i+1,\dots,b_i\} : 1 \leq i \leq r\}$; these sets are the sets of labels on the North steps in a fixed row of the lattice path diagram. A lattice path matroid is a matroid that is isomorphic to some such matroid $M[P,Q]$.

To extend this construction to delta-matroids, we take regions that are bounded by a pair of lattice paths as Figure 3 illustrates. Specifically, we have four lattice points $s_P = (0,0)$, $s_Q = (-d,d)$, $t_Q = (u,v)$, and $t_P = (u+c, v-c)$ where $v - c \geq d$ and $u,c,d \geq 0$, and two lattice paths, P from s_P to t_P, and Q from s_Q to t_Q, with P never crossing Q. These two paths, the line through s_P and s_Q, and that through t_P and t_Q, bound a region in $\mathbb{R}^2$, which we denote by $\mathcal{R}$. Label the lattice points between s_P and s_Q as shown, and do likewise for those between t_P and t_Q. We label each north step in $\mathcal{R}$ from 1 to $u+v$ according to the sum of the coordinates of its higher endpoint, and we let E be the set $\{1,2,\dots,u+v\}$ of all such labels. Let $\mathcal{P}$ be the set of lattice paths from some s_i to some t_j that remain in $\mathcal{R}$. With each path $P' \in \mathcal{P}$, let $b(P')$ be the set of labels on its north steps. The set

$$\{b(P') : P' \in \mathcal{P} \text{ going from } s_Q \text{ to } t_P\}$$

is the set of bases of a lattice path matroid on E, which we denote by $M(\mathcal{R}_{\min})$. Likewise,

$$\{b(P') : P' \in \mathcal{P} \text{ going from } s_P \text{ to } t_Q\}$$

is the set of bases of a lattice path matroid on E, which we denote by $M(\mathcal{R}_{\max})$. Below we show that $M(\mathcal{R}_{\min})$ is a quotient of $M(\mathcal{R}_{\max})$ and that the sets $b(P')$, over all $P' \in \mathcal{R}$, are the feasible sets of the full Higgs lift delta-matroid for this pair of matroids. It is not hard to check that there is no region $\mathcal{R}$ for which $M(\mathcal{R}_{\max})$ and $M(\mathcal{R}_{\min})$ are isomorphic to the two matroids in Figure 2. Thus, this construction does not yield all quotient-lift pairs of lattice path matroids.

Proposition 4.1. *With the notation above,*

(1) *$M(\mathcal{R}_{\min})$ is a quotient of $M(\mathcal{R}_{\max})$, and*
(2) *the map $P' \mapsto b(P')$ is a surjection from $\mathcal{P}$ onto the set of feasible sets of the full Higgs lift delta-matroid of the pair $(M(\mathcal{R}_{\min}), M(\mathcal{R}_{\max}))$.*

Proof. Let B be a basis of $M(\mathcal{R}_{\max})$. Fix e in $E - B$. We will verify the condition in part (4) of Lemma 2.2. View B as a lattice path, say $B = b(P_B)$. To get the required basis B' of $M(\mathcal{R}_{\min})$ (viewed as a lattice path, $P_{B'}$), take east steps from s_Q until P_B is reached, then follow P_B until a final sequence of east steps goes directly to t_P. (See Figure 4.) Assume that $f \in B'$ and $(B' \cup e) - f$ is a basis of $M(\mathcal{R}_{\min})$. Note that paths P_B and $P_{B'}$ share step f. Figure 5 compares the paths that correspond to B' and

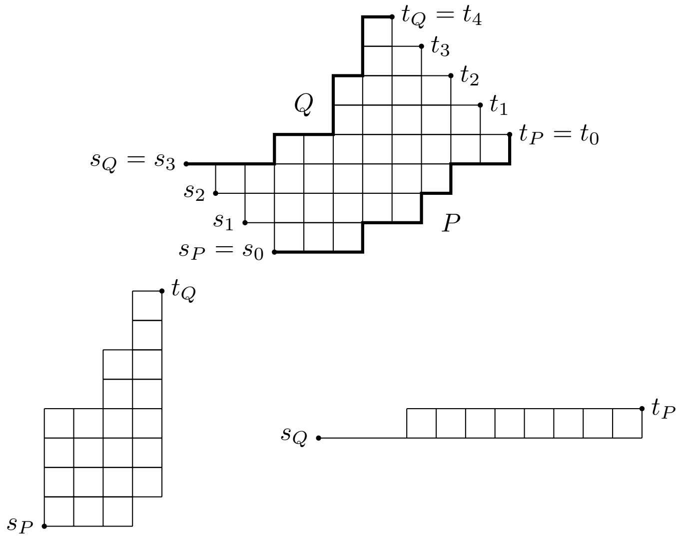


FIGURE 3. Above, a typical region of interest. Below, the lattice path representations of the two associated lattice path matroids, $M(\mathcal{R}_{\max})$ and $M(\mathcal{R}_{\min})$.

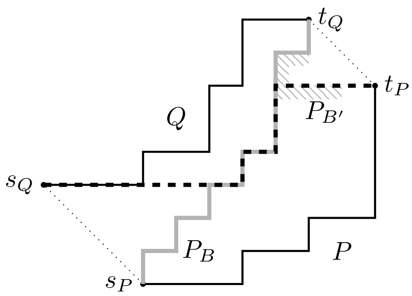


FIGURE 4. A sketch of how to get the path $P_{B'}$ (dashed) from P_B (in gray) in the proof of Proposition 4.1.

$(B' \cup e) - f$. It follows that if P_B and $P_{B'}$ share step e, then since the path corresponding to $(B' \cup e) - f$ stays in $\mathcal{R}$, and between steps e and f the paths that correspond to $(B' \cup e) - f$ and $(B \cup e) - f$ are identical, we have $(B \cup e) - f \in \mathcal{P}$. If P_B and $P_{B'}$ do not share step e, then we may assume by symmetry that e is after the last step that P_B and $P_{B'}$ share. In this case the modifications of P_B and $P_{B'}$ to get the paths for $(B \cup e) - f$ and $(B' \cup e) - f$ differ just in the sort of regions that are shaded with hatch lines in Figure 4, which are in $\mathcal{R}$. Thus, these paths stay in $\mathcal{R}$, so $(B \cup e) - f \in \mathcal{P}$ and assertion (1) holds.

For part (2), consider a path $P' \in \mathcal{P}$, say from s_u to t_v, as in Figure 6. A subpath of P' goes from a point with the same y-coordinate as s_Q to one with the same y-coordinate as t_P, and the set of labels on the north steps in that subpath is clearly a basis of $M(\mathcal{R}_{\min})$. Figure 6 shows how to create a path P'' from s_P to t_Q with $b(P') \subseteq b(P'')$. Thus, for each path P' in $\mathcal{P}$, the set $b(P')$ is a basis of a Higgs lift of $M(\mathcal{R}_{\min})$ to $M(\mathcal{R}_{\max})$.

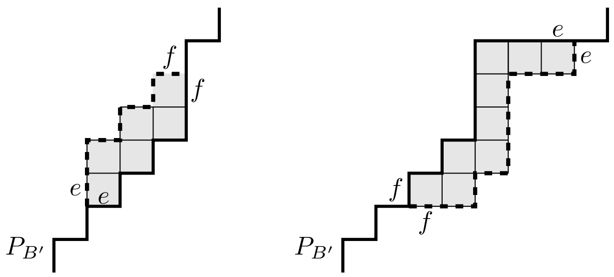


FIGURE 5. Exchanging f for a smaller element e diverts the solid path around the shaded region to the left, as the dashed path in the first part shows. Exchanging f for a larger element e diverts the path around the shaded region to the right, as the dashed path in the second part shows.

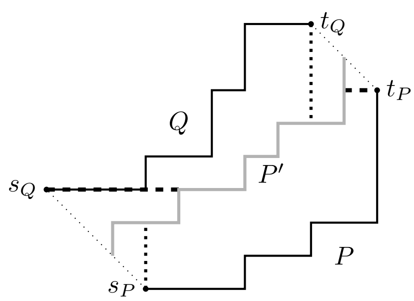


FIGURE 6. The gray line is a path P' from s_u to t_v. The dashed lines show that $b(P')$ contains a basis of $M(\mathcal{R}_{\min})$. The dotted lines show that $b(P')$ is contained in a basis of $M(\mathcal{R}_{\max})$.

We turn to the converse, showing that each basis B of each Higgs lift of $M(\mathcal{R}_{\min})$ to $M(\mathcal{R}_{\max})$ is $b(P')$ for some $P' \in \mathcal{P}$, that is, if B_0 is a basis of $M(\mathcal{R}_{\min})$ and B_1 is a basis of $M(\mathcal{R}_{\max})$, and if $B_0 \subseteq B \subseteq B_1$, then $B = b(P')$ for some path $P' \in \mathcal{P}$. We induct on $|B_1 - B|$. The base case, $B = B_1$, is obvious, so assume that $|B_1 - B| > 0$ and that the assertion holds for all diagrams $\mathcal{R}'$ and triples $B'_0 \subseteq B' \subseteq B'_1$ where B'_0 is a basis of $M(\mathcal{R}'_{\min})$ and B'_1 is a basis of $M(\mathcal{R}'_{\max})$ and $|B'_1 - B'| < |B_1 - B|$.

Let I_1 be the interval of labels on the lowest row of north steps in $\mathcal{R}$, and likewise for successive rows. We call an interval I_j lower, middle, or upper according to whether the corresponding row is below s_Q, between s_Q and t_P, or above t_P. Let P_{B_1} be the path with $b(P_{B_1}) = B_1$. We call an interval good if the north step that P_{B_1} uses in it is in B; otherwise it is bad. Since $|B_1 - B| > 0$, there is at least one bad interval.

First assume that there is a bad lower interval, say I_h. Let the north step that P_{B_1} uses in I_h be labeled x, so $x \in B_1 - B$. Each lower interval properly contains those below it, so if we delete interval I_1 from the diagram (adjusting P and s_P accordingly) to get a region $\mathcal{R}'$, then $B_1 - x$ is a basis of $M(\mathcal{R}'_{\max})$ and the induction hypothesis applies to $\mathcal{R}'$, B, and $B_1 - x$ since $|(B_1 - x) - B| < |B_1 - B|$. (The path that corresponds to $B_1 - x$ is obtained from P_{B_1} by moving each step before x northwest and changing x to an east step, as shown in Figure 7, so the path remains in $\mathcal{R}'$). By induction there is a path P' in $\mathcal{R}'$ with $b(P') = B$, and since $\mathcal{R}$ contains $\mathcal{R}'$, this path P' is also a path in $\mathcal{R}$, as we needed.

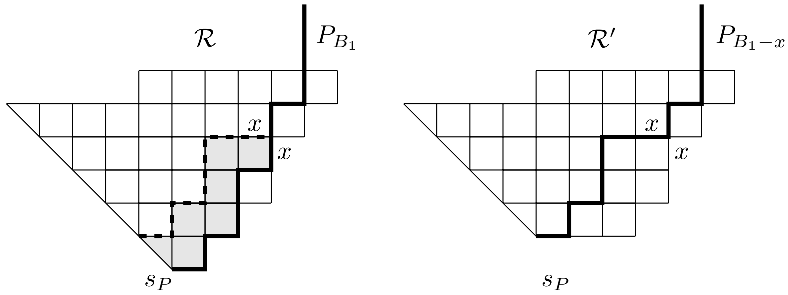


FIGURE 7. To treat a bad lower interval, replace the path P_{B_1}, shown in bold on the left, with P_{B_1-x}. Only the lower rows of the diagrams are shown.

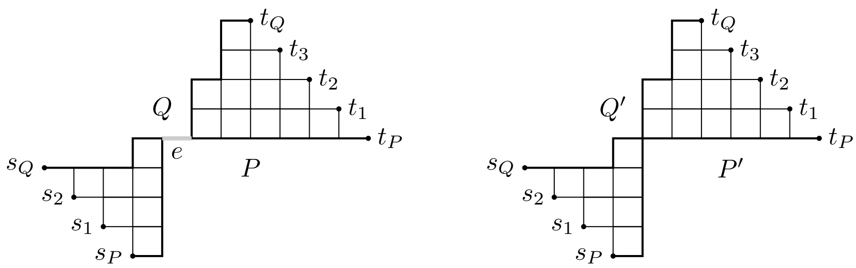


FIGURE 8. Deleting or contracting a loop, e (in gray).

We can treat bad upper intervals similarly (deleting the top interval), so now assume that the only bad intervals are middle intervals. When there are at least two bad middle intervals, we choose which to process as follows. Let I_j and I_k be the lowest and highest such intervals, respectively. Let P_{B_0} be the path with $b(P_{B_0}) = B_0$. Let the north step that P_{B_1} uses in I_j be x, so $x \in B_1 - B$, and let the north step that P_{B_0} uses in I_j be y, so $y \in B_0$, so $y \neq x$. Let x' and y' be the elements of $B_1 - B$ and B_0, respectively, defined in the same way using I_k. We cannot have both $x < y$ and $y' < x'$ since $B_0 \subseteq B_1$ and since P_{B_1} and P_{B_0} use exactly one north step from each of $I_j, I_{j+1}, \ldots, I_k$. Now assume $y < x$. (The case of $x' < y'$ is handled similarly, working with the intervals above I_k.) Let I_h be the lowest middle interval, and let $x_{h-1} < x_h < \cdots < x_j = x$ be the elements of B_1 that P_{B_1} uses as north steps in $I_{h-1}, I_h, \ldots, I_j$. Likewise, let $y_h < y_{h+1} < \cdots < y_j = y$ be the elements of B_0 that P_{B_0} uses as north steps in $I_h, I_{h+1}, \ldots, I_j$. Since $B_0 \subseteq B_1$, from $y_j < x_j$, we get $y_i \leq x_{i-1} < x_i$ for all i with $h \leq i \leq j$; thus, $x_{i-1} \in I_i$. From this, it is easy to see that if we delete the interval I_1 from the diagram to get a region $\mathcal{R}'$, then, as in the case we treated above, the induction hypothesis applies to $\mathcal{R}'$, B, and $B_1 - x$, and yields the path P' in $\mathcal{R}$ that we needed. $\square$

We call the delta-matroids constructed above, and delta-matroids that are isomorphic to them, *lattice path delta-matroids*.

Proposition 4.2. *The class of lattice path delta-matroids is closed under duals and minors.*

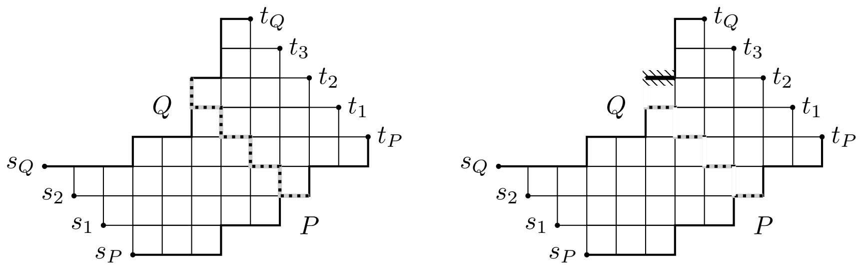


FIGURE 9. The dotted line shows the steps that are labelled e.

Proof. Dual-closure is seen by flipping the diagram around the line $y = x$. For minors, first note that a loop in a lattice path delta-matroid is represented by an east step that is in both bounding paths (thus pinching the paths together for at least that step and giving a direct sum decomposition). The deletion and contraction of a loop is obtained by eliminating this step and moving the right side of the diagram one unit to the left, as Figure 8 illustrates. The identification and treatment of coloops follows by duality. Now assume that e is neither a loop nor a coloop, so e is represented by both north and east steps, indeed, by all of the north and east steps that are at distance e from the initial steps, as Figure 9 shows. To delete e, we must use only such steps that go east, so erase those that go north, as the second part of Figure 9 shows. As shown there (highlighted with hatch lines), some steps may no longer be reached; erase them. Now shrink the east steps labelled e to points to obtain a lattice path representation of the deletion of e. Contractions are handled dually. □

With Proposition 3.1, we can strengthen Proposition 4.1 in the following way.

Corollary 4.3. *With the notation above, let $j = r(M(\mathcal{R}_{\min}))$ and $k = r(M(\mathcal{R}_{\max}))$. Fix a subset K of $\{j, j+1, \ldots, k\}$ for which $\{j, j+1, \ldots, k\} - K$ contains no pair of consecutive integers. Then $\{b(P) : P \in \mathcal{P}$ and $|b(P)| \in K\}$ is the set of feasible sets of a delta-matroid.*

We note that while $M(\mathcal{R}_{\min})$ and $M(\mathcal{R}_{\max})$ are lattice path matroids, the other Higgs lifts of $M(\mathcal{R}_{\min})$ toward $M(\mathcal{R}_{\max})$ might not be; they are in the larger class of multi-path matroids [4].

5. THE EXCLUDED-MINOR CHARACTERIZATION OF DELTA-MATROIDS

Delta-matroids form a minor-closed class of set systems. In this section, we determine the excluded minors that characterize this minor-closed class. We also prove Theorem 3.4.

The following set systems play many roles in the rest of this paper. Let

$$S_i = (\{e_1, e_2, \ldots, e_i\}, \{\emptyset, \{e_1, e_2, \ldots, e_i\}\}).$$

Let $\mathcal{S}$ be the set of all twists of the set systems in $\{S_3, S_4, \ldots\}$. Let

- $T_1 = (\{a, b, c\}, \{\emptyset, \{a, b\}, \{a, b, c\}\})$;
- $T_2 = (\{a, b, c\}, \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\})$;
- $T_3 = (\{a, b, c\}, \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\})$;
- $T_4 = (\{a, b, c\}, \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\})$;
- $T_5 = (\{a, b, c, d\}, \{\emptyset, \{a, b\}, \{a, b, c, d\}\})$;
- $T_6 = (\{a, b, c, d\}, \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c, d\}\})$;
- $T_7 = (\{a, b, c, d\}, \{\emptyset, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c, d\}\})$;

- $T_8 = (\{a, b, c, d\}, \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c, d\}\})$.

Let $\mathcal{T}$ be the set of all twists of the set systems in $\{T_1, T_2, \dots, T_8\}$. It is easy to check that none of the set systems just defined is a delta-matroid, so none of the set systems in $\mathcal{S} \cup \mathcal{T}$ is a delta-matroid.

We first prove Theorem 3.4. For that, it is useful to note that, up to isomorphism, there is only one four-element even set system $S = (E, \mathcal{F})$ such that $\emptyset, E \in \mathcal{F}$ and S is not among $U_3, U_4, \dots, U_7, T_5, T_6, T_7, T_7^*$, and S_4. Its feasible sets are all sets of even cardinality, that is, S is the even Higgs lift delta-matroid of the pair $((E, \{\emptyset\}), (E, \{E\}))$. Note that this proof does not use the fact that the class of Higgs lift delta-matroids is minor-closed; that is, instead, a corollary of the proof.

Proof of Theorem 3.4. Suppose first that a delta-matroid $D = (E, \mathcal{F})$ has a minor D' that is isomorphic to one of $U_1, U_2, \dots, U_7$. Using Lemma 2.1 and relabeling, we may assume that $D' = U_i = D\backslash X/Y$, and its collection of feasible sets is

$$\{F - Y : F \in \mathcal{F} \text{ and } Y \subseteq F \subseteq E - X\}.$$

Since $\emptyset$ and $E(U_i)$ are in $\mathcal{F}(U_i)$, the sets Y and $E - X$ are in $\mathcal{F}$, so by Lemma 1.2 there are sets $A \in \mathcal{B}(D_{\min})$ and $B \in \mathcal{B}(D_{\max})$ with $A \subseteq Y$ and $E - X \subseteq B$. The delta-matroid U_i has sets C and C' where $|C| = |C'|$, yet only one of C and C' is feasible. It follows that $A \subseteq Y \cup C \subseteq B$ and $A \subseteq Y \cup C' \subseteq B$, yet only one of $Y \cup C$ and $Y \cup C'$ is feasible in D, so D is not a Higgs lift delta-matroid by Lemma 3.2, as we needed to prove.

For the remainder of the proof, we will assume that a delta-matroid $D = (E, \mathcal{F})$ is not a Higgs lift delta-matroid, and, toward deriving a contradiction, that D does not contain any minor isomorphic to a member of $\{U_1, U_2, \dots, U_7\}$. From Corollary 3.3, we know that $\mathcal{F}$ is a subset of the full Higgs lift delta-matroid of the pair $(D_{\min}, D_{\max})$. Evidently D is missing some sets whose addition would give a Higgs lift delta-matroid.

For all non-negative integers $i \leq r(D_{\max}) - r(D_{\min})$, let N_i be the set system

$$N_i = (E, \{F \in \mathcal{F} : |F| = i + r(D_{\min})\})$$

and let $H^i = H^i_{D_{\min}, D_{\max}}$. Since D is not a Higgs lift delta-matroid, there is some proper set system N_k such that $N_k \neq H^k$. Let k be least with this property. Thus, $0 < k < r(D_{\max}) - r(D_{\min})$. From any basis of the matroid H^k we can obtain any other basis of H^k by a sequence of single-element exchanges. Also, all feasible sets in N_k are bases of H^k but not conversely. It follows that there are sets $Y \in \mathcal{F}(N_k)$ and $X \in \mathcal{B}(H^k) - \mathcal{F}(N_k)$ with $|X \triangle Y| = 2$. Let $X \triangle Y = \{x, y\}$, where $X = A \cup x$ and $Y = A \cup y$. Since $X, Y \in \mathcal{B}(H^k)$, Lemma 3.2 implies that both are spanning in $D_{\min}$ and independent in $D_{\max}$. Furthermore, there exist sets $F_x \in \mathcal{B}(D_{\min})$ and $G_x \in \mathcal{B}(D_{\max})$ such that $F_x \subseteq X \subseteq G_x$.

We show that

3.4.1. (1) *$A \cup \{x, y, z\} \in \mathcal{F}$ for some element $z \in E - A$, where z may be x; and*
(2) *$A \in \mathcal{F}$ or $A - a \in \mathcal{F}$ for some element $a \in A$.*

By applying Axiom (SE) to $(A \cup y, G_x, x)$, we find that $A \cup \{x, y\}$, $A \cup x$, or $A \cup \{x, y, z\}$ is in $\mathcal{F}$ for some element $z \in E - A$. Since $A \cup x \notin \mathcal{F}$, part (1) follows. By applying Axiom (SE) to $(A \cup y, F_x, y)$, we find that $A, A \cup x$, or $A - a$ is in $\mathcal{F}$, for some element $a \in A$. Since $A \cup x \notin \mathcal{F}$, part (2) follows.

Next, we show that

3.4.2. $A \notin \mathcal{F}$.

Suppose $A \in \mathcal{F}$. Since $A \cup x \notin \mathcal{F}$ and $A \cup y \in \mathcal{F}$, and $(D/A)|\{x, y\}$ is not isomorphic to U_1, we know that $A \cup \{x, y\} \notin \mathcal{F}$. By 3.4.1(1), $A \cup \{x, y, z\} \in \mathcal{F}$ for some $z \in E - (A \cup \{x, y\})$. Let $D' = (E', \mathcal{F}') = (D/A)|\{x, y, z\}$. Then $\mathcal{F}'$ contains $\emptyset, \{y\}$, and $\{x, y, z\}$, and avoids $\{x\}$ and $\{x, y\}$. Since D'/y is not isomorphic to U_1, $\{y, z\} \notin \mathcal{F}'$. By Axiom (SE) applied to $(\emptyset, \{x, y, z\}, x)$, we find that $\{x\}, \{x, y\}$, or $\{x, z\}$ is in $\mathcal{F}'$. Hence $\{x, z\} \in \mathcal{F}'$. Since we avoid a U_2-minor, it must be the case that the last possible feasible set, $\{z\}$, is in $\mathcal{F}'$. Now $D' \backslash y$ is isomorphic to U_1, a contradiction. Thus 3.4.2 holds.

By 3.4.1(2) and 3.4.2, we know that $A \notin \mathcal{F}$ and $A - a \in \mathcal{F}$ for some element $a \in A$. The minimality of k and having $A \in \mathcal{B}(H^{k-1}) - \mathcal{F}(N_{k-1})$ imply that N_{k-1} is not proper. Hence no set in $\mathcal{F}$ has cardinality $|A|$. If $A \cup \{x, y\} \in \mathcal{F}$, then Axiom (SE) applied to $(A \cup \{x, y\}, A - a, y)$ implies that some set in $\{A \cup x, A, (A \cup x) - a\}$ is in $\mathcal{F}$. The cardinality of the last two sets is equal to $|A|$, so neither of these is in $\mathcal{F}$, and the first also does not occur. Hence $A \cup \{x, y\} \notin \mathcal{F}$. By 3.4.1(1), $A \cup \{x, y, z\} \in \mathcal{F}$ for some $z \in E - (A \cup \{x, y\})$.

Let $D' = (E', \mathcal{F}') = (D/(A-a))|\{a, x, y, z\}$. We know that $\mathcal{F}'$ contains $\emptyset, \{a, y\}$, and $\{a, x, y, z\}$, and avoids $\{a, x\}$ and $\{a, x, y\}$. Furthermore $\mathcal{F}'$ contains no single-element sets since $\mathcal{F}$ contains no sets of cardinality $|A|$. As $D'/\{a, y\}$ is not isomorphic to U_1, $\{a, y, z\} \notin \mathcal{F}'$. If $\{a, x, z\} \in \mathcal{F}'$, then Axiom (SE) applied to $(\{a, x, z\}, \emptyset, z)$ implies that a set in $\{\{a, x\}, \{a\}, \{x\}\}$ is in $\mathcal{F}'$, a contradiction. If D' is even, then it is straightforward to check that it is isomorphic to a set system in $\{U_3, U_4, \ldots, U_7, T_5, T_6, T_7, T_7^*\}$, a contradiction. We have ruled out all singleton sets and all three-element sets from being in $\mathcal{F}'$ except possibly $\{x, y, z\}$. Hence $\{x, y, z\} \in \mathcal{F}'$. Now Axiom (SE) applied to $(\{x, y, z\}, \emptyset, z)$ implies that some set in $\{\{x, y\}, \{x\}, \{y\}\}$ is in $\mathcal{F}'$. Hence $\{x, y\} \in \mathcal{F}'$ and $D'/\{x, y\}$ is isomorphic to U_1, a contradiction. □

Next we prove the following excluded-minor characterization of delta-matroids.

Theorem 5.1. *A proper set system S is a delta-matroid if and only if S has no minor isomorphic to a set system in $\mathcal{S} \cup \mathcal{T}$.*

Recall from Corollary 3.3 that any delta-matroid may be obtained from a full Higgs lift delta-matroid by removing some feasible sets. Theorem 5.1 identifies those intervals that we must not create when removing feasible sets from Higgs lift delta-matroids in order to get general delta-matroids. We note that $\mathcal{T}$ contains 51 set systems, which are all shown in Tables 1–8 in the appendix, Section 7. We will exploit Theorem 5.1 and these tables in Section 6, where we consider delta-matroids that are built from matroids.

Proof of Theorem 5.1. Every minor of a delta-matroid is a delta-matroid. Therefore no delta-matroid has any minor in $\mathcal{S} \cup \mathcal{T}$.

Suppose that a proper set system $S = (E, \mathcal{F})$ is an excluded minor for the class of delta-matroids. Then it is not a delta-matroid but every minor of S, other than S itself, is a delta-matroid. Take sets A and B in $\mathcal{F}$ and element a in $A \triangle B$ such that $A \triangle \{a, x\}$ is not in $\mathcal{F}$ for all $x \in A \triangle B$. We assume that $|A \triangle B|$ is minimized fitting this condition. Up to taking partial duals of S, we may assume that $B \subset A$. By deleting the elements in $E - A$ and contracting the elements in B, we get a minor of S that also fails to be a delta-matroid, since Axiom (SE) fails for the triple $(A - B, \emptyset, a)$. Thus, we can take $E = A$ and $B = \emptyset$. Then $a \in A$, and $A - \{a, x\} \notin \mathcal{F}$ for all $x \in A$. Thus $|A| \geq 3$.

Suppose $A - x$ is in $\mathcal{F}$ for some element $x \in A$. Clearly $x \neq a$. By minimality of $|A \triangle B|$, Axiom (SE) applied to the triple $(A - x, \emptyset, a)$ implies that $(A - x) \triangle \{a, y\} \in \mathcal{F}$ for some element $y \in A - x$. As $A - \{a, x\}$ is not in $\mathcal{F}$, we know that $y \notin \{x, a\}$, and $A - \{a, x, y\}$ is in $\mathcal{F}$ and has three elements fewer than A. Furthermore, Axiom (SE) fails for $(A, A - \{a, x, y\}, a)$. By the minimality of $A \triangle B$, we deduce that $B = A - \{a, x, y\}$, so $|A| = 3$. Without loss of generality, $A = \{a, b, c\}$ and $x = c$, so $\{a, b\} \in \mathcal{F}$. Then $\mathcal{F}$ contains $A, \{a, b\}, \emptyset$, and some sets in $\{\{a\}, \{a, c\}\}$. It follows that S is one of T_1, T_2, T_3, or T_4.

We assume then that for all $x \in A$, the set $A - x$ is not in $\mathcal{F}$. Suppose that $A - \{x, y\}$ is in $\mathcal{F}$ for some $x, y \in A$. Clearly $x \neq y$ and $a \notin \{x, y\}$. Then by minimality of $|A \triangle B|$, Axiom (SE) applied to $(A - \{x, y\}, \emptyset, a)$ implies that there is an element $z \in A - \{x, y\}$ such that $(A - \{x, y\}) - \{a, z\}$ is in $\mathcal{F}$. Now Axiom (SE) does not hold for the triple $(A, A - \{a, x, y, z\}, a)$, since, for any element e in $\{a, x, y, z\}$, the set $A - \{a, e\}$ is not in $\mathcal{F}$. Thus $|A| \leq 4$. If $|A| < 4$, then $|A| = 3$, and it is straightforward to check that S is isomorphic to T_1^*. We assume therefore that $|A| = 4$, and $A = \{a, b, c, d\}$. Without loss of generality, $\{x, y\} = \{c, d\}$, so $\{a, b\} \in \mathcal{F}$. Now $\mathcal{F}$ does not contain any three-element sets, nor does it contain $\{b, c\}, \{b, d\}$, or $\{c, d\}$. By the minimality of $|A \triangle B|$, Axiom (SE) holds for each triple containing two sets in $\mathcal{F}$ and an element in their symmetric difference unless the two sets are A and B. If $\{w\} \in \mathcal{F}$ for some $w \in \{b, c, d\}$, then there is an element $v \in \{a, b, c, d\} \triangle \{w\}$ such that $\{a, b, c, d\} \triangle \{a, v\}$ is in $\mathcal{F}$. As no such set is in $\mathcal{F}$, we know that $\{a\}$ is the only possible singleton set in $\mathcal{F}$. Therefore, $\mathcal{F}$ contains $A, \{a, b\}, \emptyset$, and some sets in $\{\{a, c\}, \{a, d\}, \{a\}\}$. It is straightforward to check that either S is isomorphic to one of T_5, T_6, T_7, or T_8, or S/a is isomorphic to T_1^* or T_2^*.

We may now assume that $A - x$ and $A - \{x, y\}$ are not in $\mathcal{F}$ for all $x, y \in A$. Let A' be the second largest set in $\mathcal{F}$. Then Axiom (SE) fails for the triple (A, A', e), for any $e \in A - A'$. Hence $|A'| = |B| = 0$, by minimality of $|A \triangle B|$. Let $|A| = k$. Clearly $k \geq 3$. Then $S \cong S_k$. $\square$

The next two results are easily obtained from Theorem 5.1. Both characterize even delta-matroids. Let $\mathcal{T}_{5,6,7}$ be the set of all set systems that are twists of T_5, T_6, or T_7.

Corollary 5.2. *A proper, even set system S is an even delta-matroid if and only if S has no minor isomorphic to a set system in $\{(E, \mathcal{F}) \in \mathcal{S} : |E|$ is even$\} \cup \mathcal{T}_{5,6,7}$.*

The second uses a result of Bouchet [9, Lemma 5.4]: within the class of delta-matroids, S_1 is the unique excluded minor for even delta-matroids. Moreover each set system in $\mathcal{T} - \mathcal{T}_{5,6,7}$ has a minor isomorphic to S_1. Adding S_1 to the list of minors to avoid therefore eliminates the need to require that S be even.

Corollary 5.3. *A proper set system S is an even delta-matroid if and only if S has no minor isomorphic to a set system in $\{S_1\} \cup \mathcal{S} \cup \mathcal{T}_{5,6,7}$.*

A delta-matroid is a matroid exactly when its feasible sets are equicardinal, so it is straightforward to determine the excluded minors for matroids from Theorem 5.1.

Corollary 5.4. *A proper set system $S = (E, \mathcal{F})$ is a matroid if and only if all of the sets in $\mathcal{F}$ have the same size, and S has no minor isomorphic to a set system in*

$$\{T_5 * \{a, c\},\ T_6 * \{a, d\}\} \cup \{S_{2k} * \{e_1, e_2, \dots, e_k\} : k \geq 2\}.$$

Excluded-minor characterizations for a number of minor-closed classes of matroids are known. For a minor-closed class of matroids $\mathcal{M}$, let $\mathrm{Ex}(\mathcal{M})$ be its set of excluded minors. The next corollary follows immediately from Corollary 5.4.

Corollary 5.5. *For a minor-closed class of matroids $\mathcal{M}$, a proper set system $S = (E, \mathcal{F})$ is in $\mathcal{M}$ if and only if all of the sets in $\mathcal{F}$ have the same size and S has no minor isomorphic to a set system in*

$$\mathrm{Ex}(\mathcal{M}) \cup \{T_5 * \{a, c\},\ T_6 * \{a, d\}\} \cup \{S_{2k} * \{e_1, e_2, \dots, e_k\} : k \geq 2\}.$$

Let $\mathbb{F}$ be a finite field. For a finite set E, let C be a skew-symmetric $|E|$ by $|E|$ matrix over $\mathbb{F}$, with rows and columns indexed by the elements of E. Thus, the diagonal of C can be non-zero only when $\mathbb{F}$ has characteristic two. Let $C\,[A]$ be the principal submatrix of C induced by the set $A \subseteq E$. Bouchet showed in [10] that we obtain a delta-matroid, denoted $D(C)$, with ground set E by taking as the feasible sets all $A \subseteq E$ such that the rank of the matrix $C[A]$ is $|A|$. A delta-matroid is called *representable over* $\mathbb{F}$ if it has a twist that is isomorphic to $D(C)$ for some skew-symmetric matrix C. Note that the empty set is feasible in $D(C)$. Thus, for a delta-matroid D, if $D_{\min} \neq (E, \{\emptyset\})$, then D does not have a matrix representation. However, every delta-matroid has a partial dual that has the empty set as a feasible set; simply take the twist on any feasible set. In particular, any matroid M with rank exceeding zero does not have the empty set as a basis, but, for any basis B of M, the delta-matroid $M * B$ has the empty set among its feasible sets. The following result by Bouchet [10] shows that, as one would infer from the common terminology, delta-matroid representability agrees with matroid representability on the class of matroids.

Proposition 5.6. *A matroid representable over a field $\mathbb{F}$ is also representable over $\mathbb{F}$ as a delta-matroid.*

To be explicit, suppose that a matroid M is representable over $\mathbb{F}$ and that B is a basis of M. Then M has a representation of the form $(I|A)$ where I is a $|B| \times |B|$ identity matrix and the columns of I correspond to the elements of B. It is not difficult to see that if

$$C = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix},$$

then $M * B = D(C)$.

A delta-matroid representable over the field with two elements is called *binary*. Let

- $P_1 = (\{a, b, c\}, \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\})$;
- $P_2 = (\{a, b, c\}, \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\})$;
- $P_3 = (\{a, b, c\}, \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\})$;
- $P_4 = (\{a, b, c, d\}, \{\emptyset, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\})$;
- $P_5 = (\{a, b, c, d\}, \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c, d\}\})$.

Let $\mathcal{P}$ be the set of all twists of P_1, P_2, P_3, P_4, or P_5. In [12], Bouchet and Duchamp proved the following theorem.

Theorem 5.7. *A delta-matroid is a binary delta-matroid if and only if it has no minor isomorphic to a delta-matroid in $\mathcal{P}$.*

It is worth noting that $P_5 * \{a, c\} \cong U_{2,4}$. Thus the unique excluded minor for binary matroids is in $\mathcal{P}$, as one would expect. Combining Theorem 5.1 with Bouchet's characterization gives the following corollary.

Corollary 5.8. *A proper set system S is a binary delta-matroid if and only if S has no minor isomorphic to a set system in $\mathcal{P} \cup \mathcal{S} \cup \mathcal{T}$.*

6. Matroid stack delta-matroids

In Section 3, we found that the collection of bases of the Higgs lifts between a quotient of a matroid M and M, or of an appropriately chosen subcollection of these Higgs lifts, gives a delta-matroid. In Section 4, we considered Higgs lifts between particular pairs of lattice path matroids. It is natural to ask, more generally, when a set of matroids can form the layers of a delta-matroid. More precisely, suppose we take matroids $M_1, M_2, \ldots, M_n$ on E where $1 \leq r(M_{i+1}) - r(M_i) \leq 2$ for all $i \in \{1, 2, \ldots, n-1\}$. Under what circumstances is the set system $\big(E, \mathcal{B}(M_1) \cup \mathcal{B}(M_2) \cup \cdots \cup \mathcal{B}(M_n)\big)$ a delta-matroid? This is what we explore in this section.

Let $S = (E, \mathcal{F})$ be a proper set system where the smallest sets in $\mathcal{F}$ have cardinality k and the largest have cardinality ℓ. Let N_i be the set system $(E, \{F : |F| = i \text{ and } F \in \mathcal{F}\})$ for $i \in \{k, k+1, \dots, \ell\}$. We say that $N_k, N_{k+1}, \dots, N_\ell$ is the *stack of* S. If, for some i between k and ℓ, no sets in $\mathcal{F}$ have size i, then $N_i = (E, \emptyset)$, which is not proper. If every proper set system in the stack of S is a matroid, then we say that S is a *matroid stack set system*. Furthermore, if S is a delta-matroid, then we say that S is a *matroid stack delta-matroid*. Since the dual of a matroid is a matroid, it follows that the dual of a matroid stack set system is also a matroid stack set system, and likewise for a matroid stack delta-matroid.

We show that if the matroids in the stack of a matroid stack set system S all belong to a minor-closed class $\mathcal{M}$, then the proper set systems in the stack of any minor of S all belong to $\mathcal{M}$. In particular, this implies that the class of matroid stack delta-matroids is closed under taking minors.

Lemma 6.1. *Let $\mathcal{M}$ be a minor-closed class of matroids. Let $S = (E, \mathcal{F})$ be a matroid stack set system where the matroids in the stack of S are in $\mathcal{M}$. If S' is a minor of S, then S' is a matroid stack set-system, and the matroids in the stack of S' are all in $\mathcal{M}$.*

Proof. Take $e \in E$. It suffices to show that all of the proper set systems in the stack of $S\backslash e$ and S/e are matroids in $\mathcal{M}$. We consider $S\backslash e$ first. If e is a coloop of S, then e is a coloop of every matroid in the stack of S, and the result is clear. So assume that e is not a coloop. Let $N = (E, \mathcal{F}')$ be a proper set system in the stack of $S\backslash e$. The sets in $\mathcal{F}'$ are equicardinal feasible sets in $\mathcal{F}(S)$ that avoid e, so S has a matroid M in its stack such that $\mathcal{F}' \subseteq \mathcal{B}(M)$. Furthermore, the sets in $\mathcal{F}'$ are exactly the bases of M that avoid e, so $N = M\backslash e$. Thus $N \in \mathcal{M}$.

Let $\mathcal{M}^* = \{M^* : M \in \mathcal{M}\}$. Then $\mathcal{M}^*$ is a minor-closed class of matroids. Note that S^* is a matroid stack for which each proper set system in the stack belongs to $\mathcal{M}^*$. Hence the stack of $S^*\backslash e$ has all of its proper set systems in $\mathcal{M}^*$, and so $(S^*\backslash e)^*$ is a matroid stack for which each proper set system in the stack belongs to $\mathcal{M}$. This last set system is equal to S/e. $\square$

In the next corollary, we use Theorem 5.1 to find the excluded minors for matroid stack delta-matroids within the class of matroid stack set systems. The excluded minors are exactly those set systems in $\mathcal{S} \cup \mathcal{T}$ that are matroid stack set systems. Note that any proper set system $(E, \mathcal{F})$ where $|E| = 3$ and the sets in $\mathcal{F}$ are equicardinal is a matroid. For this reason, every twist of T_1, T_2, T_3, or T_4 is an excluded minor for matroid stack delta-matroids. Let $\mathcal{T}_{1,2,3,4}$ be the set of these twists.

Corollary 6.2. *Let D be a matroid stack set system. Then D is a matroid stack delta-matroid if and only if it contains no minor isomorphic to a set system in any of the following sets:*

(1) $\{S_k * X : k \geq 3, X \subseteq E(S_k),$ *and* $|X| \neq k/2\}$,
(2) $\mathcal{T}_{1,2,3,4}$,
(3) $\{T_5,\ T_5 * a,\ T_5 * \{b, c, d\}\}$,
(4) $\{T_6,\ T_6 * a,\ T_6 * b,\ T_6 * \{b, c, d\}\}$,
(5) $\{T_7,\ T_7 * a,\ T_7 * b,\ T_7 * \{a, c, d\},\ T_7 * \{b, c, d\},\ T_7^*\}$,
(6) $\{T_8,\ T_8 * a,\ T_8 * b,\ T_8 * \{a, c, d\},\ T_8 * \{b, c, d\},\ T_8^*\}$.

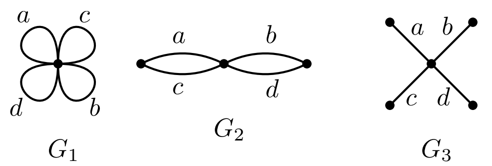


FIGURE 10. The spanning trees of these graphs are the feasible sets of P_5.

Proof. If D is a matroid stack set system that is not a delta-matroid then it must have a minor D' isomorphic to a set system in $\mathcal{S} \cup \mathcal{T}$. Moreover, Lemma 6.1 implies that D' must be a matroid stack set system. The result follows by checking which elements of $\mathcal{S} \cup \mathcal{T}$ are matroid stack systems. □

Note that representability within the stack of a matroid stack delta-matroid does not guarantee representability of the delta-matroid. For example, P_5 is an excluded minor for binary delta-matroids, but it is also a matroid stack delta-matroid where each matroid in the stack is binary. In fact, each matroid is graphic, and these graphs are depicted in Figure 10.

The class of even delta-matroids is minor-closed. Hence the next result is a corollary of Lemma 6.1.

Corollary 6.3. *The class of matroid stack delta-matroids that are even is minor-closed and dual-closed.*

The following result is easily obtained from Corollary 6.2 by identifying those set systems in the excluded minors for matroid stack delta-matroids that are even.

Corollary 6.4. *An even matroid stack set system is an even matroid stack delta-matroid if and only if it contains no minor isomorphic to a set system in one of the following sets:*

(1) $\{S_{2k} * X : k \geq 2, X \subseteq E(S_{2k}),$ *and* $|X| \neq k\}$,
(2) $\{T_5,\ T_5 * a,\ T_5 * \{b, c, d\}\}$,
(3) $\{T_6,\ T_6 * a,\ T_6 * b,\ T_6 * \{b, c, d\}\}$,
(4) $\{T_7,\ T_7 * a,\ T_7 * b,\ T_7 * \{b, c, d\},\ T_7 * \{a, c, d\},\ T_7^*\}$.

We next consider matroid stack delta-matroids where each matroid in the stack is paving. A rank-r matroid is *paving* if each of its circuits has size at least r. Although the class of paving matroids is closed under minors, it is not closed under duality. Let D be a set system where every proper set system in its stack is a paving matroid. Then we say that D is a *paving set system*. If D is also a delta-matroid, then we say that D is a *paving delta-matroid*. The next result follows from Lemma 6.1.

Corollary 6.5. *Every minor of a paving delta-matroid is a paving delta-matroid.*

By identifying the paving set systems among the excluded minors for matroid stack delta-matroids, we find the excluded minors for paving delta-matroids.

Corollary 6.6. *A paving set system is a paving delta-matroid if and only if it contains no minor isomorphic to a set system in the following sets:*

(1) $\{S_i : i \geq 3\}$,
(2) $\{T_1 * \{b, c\},\ T_1^*\}$,
(3) $\{T_2,\ T_2 * \{a, b\},\ T_2 * \{b, c\},\ T_2^*\}$,
(4) $\{T_3 * b,\ T_3 * \{b, c\}\}$,
(5) $\{T_4,\ T_4 * b,\ T_4 * \{a, c\},\ T_4 * \{b, c\}\}$,

(6) $\{T_6 * \{b, c, d\}\}$,
(7) $\{T_7,\ T_7 * b,\ T_7 * \{b, c, d\}\}$,
(8) $\{T_8,\ T_8 * \{b, c, d\}\}$.

Next we consider matroid stack delta-matroids where each matroid in the stack is a sparse paving matroid. A matroid is *sparse paving* if it is paving and its dual is paving. Equivalently, a matroid is sparse paving if each non-spanning circuit is a hyperplane. Let D be a set system where every proper set system in its stack is a sparse paving matroid. Then we say that D is a *sparse paving set system*. If D is also a delta-matroid, then we say that D is a *sparse paving delta-matroid*. It is easy to see that every minor of a sparse paving matroid is sparse paving. Note that the class of sparse paving delta-matroids is closed under duality. Hence the next result follows immediately from Lemma 6.1.

Corollary 6.7. *The class of sparse paving delta-matroids is minor-closed and dual-closed.*

In [24], it is conjectured that, asymptotically, almost all matroids are sparse paving. That is, if $sp(n)$ is the number of sparse paving matroids with n elements, and $m(n)$ is the number of matroids with n elements, then it is conjectured that $\lim_{n\to\infty} \frac{sp(n)}{m(n)} = 1$. We make the following related conjecture.

Conjecture 6.8. *Asymptotically, almost all matroid stack delta-matroids are sparse paving.*

In a similar vein, one might wonder if, asymptotically, almost all delta-matroids are sparse paving, but this is far from being true, as the number of delta-matroids is significantly greater than the number of matroid stack delta-matroids. It is shown in [20] that the number d_n of delta-matroids with ground set $\{1, \dots, n\}$ is at least $2^{2^{n-1}}$. On the other hand, in [2] it is shown that the number m_n of matroids with ground set $\{1, \dots, n\}$ satisfies $\log\log m_n \le n - \frac{3}{2}\log n + O(1)$, where all logs are taken to base 2. A crude estimate gives an upper bound of $f_n = (m_n + 1)^{n+1}$ for the number of matroid stack delta-matroids with ground set $\{1, \dots, n\}$ and $\log\log f_n = n - \frac{1}{2}\log n + O(1) < n - 1 \le \log\log d_n$.

Repeating this analysis for even delta-matroids yields a different picture, as it is also shown in [20] that the number e_n of even delta-matroids with ground set $\{1, \dots, n\}$ satisfies

$$n - 1 - \log n \le \log\log e_n \le n - \log n + O(\log\log n),$$

with the lower bound being the number of even sparse paving delta-matroids with ground set $\{1, \dots, n\}$, so we pose the following open question.

Open Question 6.9. *Asymptotically, are almost all even delta-matroids sparse paving?*

It is straightforward to identify the sparse paving set systems that are excluded minors for matroid stack delta-matroids. These comprise the excluded minors for sparse paving delta-matroids within the class of sparse paving set systems. Since the class of sparse paving delta-matroids is closed under duality, every set system in the set of excluded minors for sparse paving delta-matroids has its dual also in the list of excluded minors.

Corollary 6.10. *A sparse paving set system is a sparse paving delta-matroid if and only if it contains no minor isomorphic to a set system in*

$$\{S_i : i \ge 3\} \cup \{T_2,\ T_2^*,\ T_3 * b,\ T_4 * b,\ T_4 * \{a, c\}\}.$$

Next we consider matroid stack delta-matroids with a stack of quotients. That is, let D be a matroid stack set system. If every matroid in the stack is a quotient of the matroid with next highest rank in the stack, then we say that D is a *quotient set system*. It follows from Lemma 2.2 that a quotient of a quotient of M is also a quotient of M. Therefore every matroid in the stack of a quotient set system is a quotient of every matroid in the stack with higher rank. If D is a quotient set system, and D is a delta-matroid, then we say that it is a *quotient delta-matroid*. It also follows from Lemma 2.2 that D^* is also a quotient delta-matroid.

Lemma 6.11. *The class of quotient delta-matroids is minor-closed and dual-closed.*

Proof. Let $D = (E, \mathcal{F})$ be a quotient delta-matroid and take $e \in E$. Since $D/e = (D^*\backslash e)^*$, it suffices to show that $D\backslash e$ is a quotient delta-matroid. By Lemma 6.1, $D\backslash e$ is a matroid stack delta-matroid. The matroids M and M' in the stack of $D\backslash e$ are obtained from some matroids N and N' in the stack of D by deleting e. Without loss of generality, we assume that N is a quotient of N'. Then, by Lemma 2.3, M is a quotient of M'. $\square$

Note that if M is the matroid with ground set $\{1, 2, 3, 4\}$ and set of bases

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\},$$

then $M * \{1, 3\}$ is not a matroid stack delta-matroid. Therefore none of the classes of matroid stack delta-matroids, sparse paving delta-matroids, or quotient delta-matroids is closed under twists.

The following result is easily obtained from Corollary 6.2 by identifying the quotient set systems in $\mathcal{S} \cup \mathcal{T}$.

Corollary 6.12. *A quotient set system is a quotient delta-matroid if and only if it does not have a minor in*

$$\{S_i : i \geq 3\} \cup \{T_1,\ T_1^*,\ T_2,\ T_2^*,\ T_3,\ T_4,\ T_4^*,\ T_5,\ T_6,\ T_7,\ T_7^*,\ T_8,\ T_8^*\}.$$

We note the following two properties of even sparse paving set systems. A simple generalization of Lemma 4.1 from [20] shows that if the stack of an even sparse paving set system S contains no improper set systems other than those required to ensure evenness, then S is a delta-matroid. Moreover an even sparse paving set system is also a quotient set system. Hence we have the following proposition.

Proposition 6.13. *If S is an even sparse paving set system, then S is a quotient set system.*

7. Appendix: The twists of $T_1, T_2, \ldots, T_8$

T_1	$\emptyset$		$\{a,b\}$	$\{a,b,c\}$	$\emptyset$	$\{c\}$		$\{a,b,c\}$	T_1^*
$T_1 * \{a\}$		$\{a\}$ $\{b\}$	$\{b,c\}$			$\{a\}$	$\{b,c\}$ $\{a,c\}$		$T_1 * \{b,c\}$
$T_1 * \{c\}$		$\{c\}$	$\{a,b\}$	$\{a,b,c\}$	$\emptyset$	$\{c\}$	$\{a,b\}$		$T_1 * \{a,b\}$

TABLE 1. All twists of T_1 up to isomorphism. Dual pairs are side by side.

T_2	$\emptyset$		$\{a,b\}$ $\{a,c\}$	$\{a,b,c\}$	$\emptyset$	$\{c\}$ $\{b\}$		$\{a,b,c\}$	T_2^*
$T_2 * \{a\}$		$\{a\}$ $\{b\}$ $\{c\}$	$\{b,c\}$			$\{a\}$	$\{b,c\}$ $\{a,c\}$ $\{a,b\}$		$T_2 * \{b,c\}$
$T_2 * \{c\}$		$\{a\}$ $\{c\}$	$\{a,b\}$	$\{a,b,c\}$	$\emptyset$	$\{c\}$	$\{b,c\}$ $\{a,b\}$		$T_2 * \{a,b\}$

TABLE 2. All twists of T_2 up to isomorphism.

T_3	$\emptyset$	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$					
$T_3 * \{b\}$		$\{a\}$ $\{b\}$	$\{a,b\}$ $\{a,c\}$						
$T_3 * \{a\}$	$\emptyset$	$\{a\}$ $\{b\}$	$\{b,c\}$			$\{a\}$	$\{b,c\}$ $\{a,c\}$	$\{a,b,c\}$	$T_3 * \{b,c\}$

TABLE 3. All twists of T_3 up to isomorphism. A twist alone in a row is self-dual.

T_4	$\emptyset$	$\{a\}$	$\{a,b\}$ $\{a,c\}$	$\{a,b,c\}$	$\emptyset$	$\{c\}$ $\{b\}$	$\{b,c\}$	$\{a,b,c\}$	T_4^*
$T_4 * \{a\}$	$\emptyset$	$\{a\}$ $\{b\}$ $\{c\}$	$\{b,c\}$			$\{a\}$	$\{b,c\}$ $\{a,c\}$ $\{a,b\}$	$\{a,b,c\}$	$T_4 * \{b,c\}$
$T_4 * \{b\}$		$\{a\}$ $\{b\}$	$\{a,b\}$ $\{a,c\}$	$\{a,b,c\}$	$\emptyset$	$\{c\}$ $\{b\}$	$\{b,c\}$ $\{a,c\}$		$T_4 * \{a,c\}$

TABLE 4. All twists of T_4 up to isomorphism.

T_5	$\emptyset$	$\{a,b\}$	$\{a,b,c,d\}$
$T_5 * \{a,c\}$		$\{a,c\}$ $\{b,c\}$ $\{b,d\}$	
$T_5 * \{a\}$	$\{a\}$ $\{b\}$	$\{b,c,d\}$	
$T_5 * \{b,c,d\}$	$\{a\}$	$\{b,c,d\}$ $\{a,c,d\}$	
$T_5 * \{a,b\}$	$\emptyset$	$\{a,b\}$ $\{c,d\}$	
$T_5 * \{c,d\}$	$\{c,d\}$ $\{a,b\}$	$\{a,b,c,d\}$	

TABLE 5. All twists of T_5 up to isomorphism.

T_6	$\emptyset$	$\{a,b\}$ $\{a,c\}$	$\{a,b,c,d\}$
$T_6 * \{b\}$	$\{a\}$ $\{b\}$	$\{a,b,c\}$ $\{a,c,d\}$	
$T_6 * \{a,d\}$		$\{a,d\}$ $\{b,c\}$ $\{b,d\}$ $\{c,d\}$	
$T_6 * \{a\}$	$\{a\}$ $\{b\}$ $\{c\}$	$\{b,c,d\}$	
$T_6 * \{b,c,d\}$	$\{a\}$	$\{b,c,d\}$ $\{a,c,d\}$ $\{a,b,d\}$	
$T_6 * \{a,b\}$	$\emptyset$	$\{a,b\}$ $\{b,c\}$ $\{c,d\}$	
$T_6 * \{c,d\}$	$\{c,d\}$ $\{a,d\}$ $\{a,b\}$	$\{a,b,c,d\}$	

TABLE 6. All twists of T_6 up to isomorphism.

$\emptyset$		$\{a,b\}$ $\{a,c\}$ $\{a,d\}$		$\{a,b,c,d\}$	$\emptyset$		$\{c,d\}$ $\{b,d\}$ $\{b,c\}$		$\{a,b,c,d\}$
		T_7					T_7^*		
	$\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$		$\{b,c,d\}$			$\{a\}$		$\{b,c,d\}$ $\{a,c,d\}$ $\{a,b,d\}$ $\{a,b,c\}$	
		$T_7 * \{a\}$					$T_7 * \{b,c,d\}$		
	$\{a\}$ $\{b\}$		$\{a,b,c\}$ $\{a,b,d\}$ $\{a,c,d\}$			$\{d\}$ $\{c\}$ $\{b\}$		$\{b,c,d\}$ $\{a,c,d\}$	
		$T_7 * \{b\}$					$T_7 * \{a,c,d\}$		
$\emptyset$		$\{a,b\}$ $\{b,c\}$ $\{b,d\}$ $\{c,d\}$					$\{c,d\}$ $\{a,d\}$ $\{a,c\}$ $\{a,b\}$		$\{a,b,c,d\}$
		$T_7 * \{a,b\}$					$T_7 * \{c,d\}$		

TABLE 7. All twists of T_7 up to isomorphism.

$\emptyset$	$\{a\}$	$\{a,b\}$ $\{a,c\}$ $\{a,d\}$		$\{a,b,c,d\}$	$\emptyset$		$\{c,d\}$ $\{b,d\}$ $\{b,c\}$	$\{b,c,d\}$	$\{a,b,c,d\}$
		T_8					T_8^*		
$\emptyset$	$\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$		$\{b,c,d\}$			$\{a\}$		$\{b,c,d\}$ $\{a,c,d\}$ $\{a,b,d\}$ $\{a,b,c\}$	$\{a,b,c,d\}$
		$T_8 * \{a\}$					$T_8 * \{b,c,d\}$		
	$\{a\}$ $\{b\}$	$\{a,b\}$	$\{a,b,c\}$ $\{a,b,d\}$ $\{a,c,d\}$			$\{d\}$ $\{c\}$ $\{b\}$	$\{c,d\}$	$\{b,c,d\}$ $\{a,c,d\}$	
		$T_8 * \{b\}$					$T_8 * \{a,c,d\}$		
$\emptyset$	$\{b\}$	$\{a,b\}$ $\{b,c\}$ $\{b,d\}$ $\{c,d\}$					$\{c,d\}$ $\{a,d\}$ $\{a,c\}$ $\{a,b\}$	$\{a,c,d\}$	$\{a,b,c,d\}$
		$T_8 * \{a,b\}$					$T_8 * \{c,d\}$		

TABLE 8. All twists of T_8 up to isomorphism.

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