Volumetric Uncertainty Bounds and Optimal Configurations for Converging Beam Triple LIDAR

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Abstract

We consider the problem of quantifying uncertainty for *converging* beam triple LIDAR when the input uncertainty follows a uniform distribution. We determine expressions for the range (i.e. set of reachable points) for the reconstructed velocity vector as a function of any particular setting of the nominal input parameters and determine an explicit lower (and upper) bound on the (averaged) volume (with respect to Lebesgue measure), in $\mathbb{R}^3$, of that range. We show that the size of any such bound is inversely proportional to the absolute value of the triple scalar product of the unit vectors characterizing the Doppler measurement directions (optimized over the uncertainty region) in $\mathbb{R}^6$ associated with the nominal angle settings under consideration. This leads to the conclusion that the nominal LIDAR configurations that minimize output uncertainty ought to be those in which the value of the triple scalar product of the Doppler unit vectors is at its largest.

KEYWORDS: CONVERGING BEAM LIDAR; WIND VELOCITY MEASUREMENT; DOPPLER LIDAR; VELOCITY RECONSTRUCTION; INPUT UNCERTAINTY; UNCERTAINTY PROPAGATION; GRID SEARCH OPTIMIZATION; GRADIENT BOUNDS; HESSIAN BOUNDS

1 Introduction

It has been demonstrated that the angles subtended at the point of intersection in converging beam triple LIDAR can have an important effect on the (estimated) standard deviations associated with components of the reconstructed velocity vector in Cartesian coordinates (see Holtom & Brooms [6]). It was shown that a lower bound on the (estimated) standard deviation on each component can grow as fast as $1/|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$ as $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)| \to 0$, where $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$ is the absolute value of the triple scalar product of the three Doppler unit vectors and where $(\boldsymbol{\theta}_0, \ \boldsymbol{\phi}_0)$ holds the nominal values of the azimuthal, and elevation, angles characterizing those vectors. In view of the above fact, along with the observation that $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)| \leq 1$, then it is suggested that one avoids LIDAR placement locations in conjunction with velocity measurement positions that would cause the value of $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$ to be close to 0. Subject to the size of its numerator, the lower bound for the (estimated) standard deviation is minimized when $\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0) = 1$, which occurs when the Doppler unit vectors are perpendicular to each other. However, that fact alone does not necessarily prove that opting for a configuration in which $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$ is at its largest, would lead to values for the (estimated) standard deviations which are at their smallest.

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In order to further characterize how best to determine the LIDAR configurations that would likely lead to the smallest possible reconstructed velocity output uncertainty, we take a different (albeit complementary) approach to the one that was adopted in [6]. The basic approach used herein is to assume that the input uncertainty has compact interval support in each dimension, but then proceed to propagate that uncertainty forward (aided, where necessary, with the additional assumption of uniformity and mutual independence of the input uncertainty components) in order to yield the range of uncertainty around the nominal reconstructed Cartesian velocity vector, along with its volume.

In the case in which there is no uncertainty in the Doppler angles but uniform uncertainty in the Doppler velocities, then the output uncertainty takes the form of Lebesgue measure within a parallelepiped, whose volume can be expressed in terms of the volume of the input uncertainty range and the value of $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$. In the case where the Doppler angles are also characterized by uniform uncertainty, a lower bound on the volume of the reconstructed velocity uncertainty range, expressed in terms of the volume of the Doppler velocity range, and the value of $|\Delta(\cdot, \cdot)|$ maximized over the input uncertainty range, is presented. As an alternative, if we instead consider the volume conditioned upon the angles governing the LIDAR orientations, and then average that across the angle uncertainty range, then we are able to present both lower and upper bounds that are amenable to numerical computation.

The forward propagation of input uncertainty in order to then obtain a description of the region of output uncertainty has been considered by other authors for other physical systems under different sets of conditions. Jiang et al. [7] consider the case in which the input uncertainty range is described by an ellipsoid under the assumption that the joint distribution of the input variables adheres to Lebesgue measure: - this is viewed upon in a later paper as corresponding to a *pseudo-probability* approach to propagating uncertainty through to the output variable(s) (Liu et al. [10]). Systems in which both the input and output vectors are each known to reside within their respective hyper-rectangles, from which one then calculates the unknown output region from the known input region (the forward problem), or vice versa (the inverse problem), was mentioned in the case of the former, and explicitly considered in the latter, by Liu et al. [9]. A survey of the methods and techniques that have been used in relation to both forward, and inverse, uncertainty propagation is presented in the introduction to [11].

The rest of the paper is organized as follows. In the next section, we introduce and review the notation, geometrical set-up, variables and various quantities of interest, as well as the statistical and mathematical assumptions, that will be required in order to progress our analysis. In Section 3, we determine bounds on the volume in $\mathbb{R}^3$ that could be occupied by the reconstructed velocity vector as a result of trying to measure wind velocity at a specific position in space: this is first carried out in the case in which the *demanded* angles are not subject to any uncertainty, and then in the case in which they are subject to uncertainty. Being devoid of a closed form expression for the optimum values of $|\Delta(\cdot, \cdot)|$ within a compact domain of $\mathbb{R}^3 \times \mathbb{R}^3$, which are key for the calculation of the aforementioned bounds, we present in Section 4 a technique, based on the concept of *grid search optimization*, for either deducing a (small) under-estimate of the minimum value, or a (small) over-estimate of the maximum value, of the objective function. We demonstrate our analytical deductions in Section 5 with some numerical examples, and then summarize and close out our discussion in Section 6.

2 Mathematical Preliminaries

Throughout we shall work with the index set $\mathcal{I} = \{1, 2, 3\}$. Let $\{\mathbf{e}_i : i \in \mathcal{I}\}$ be the unit vectors for the standard basis in $\mathbb{R}^3$, where $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$, correspond to the x, y, z directions in the right-handed
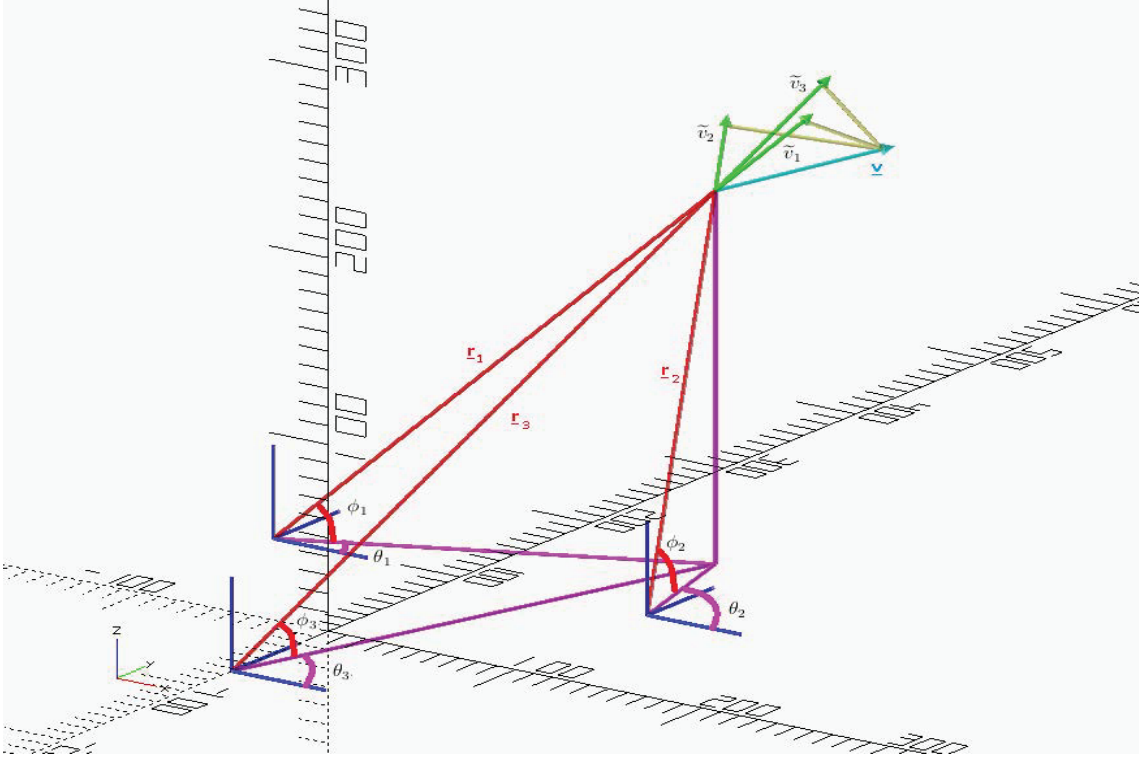
Figure 1: Geometric set-up of the convergent triple-beam LIDAR technology

Cartesian system, respectively. Both equality and inequality relationships between pairs of vectors of commensurate size are to be interpreted componentwise.

Let $\{\hat{\mathbf{r}}_i : i \in \mathcal{I}\}$ be the unit vectors corresponding to the directions of each of the laser/LIDAR beams i.e. the Doppler LIDAR basis vectors. Thus, from the point of location of each of the Doppler LIDAR beams, the point of position in space for which a velocity measurement is being sought, is represented by $\mathbf{r}_i = \widetilde{r}_i \hat{\mathbf{r}}_i$, for each lidar beam $i \in \mathcal{I}$, respectively.

A wind velocity, $\mathbf{w}$, relative to the aforementioned standard basis may represented as

$$\mathbf{w} = w_1 \mathbf{e}_1 + w_2 \mathbf{e}_2 + w_3 \mathbf{e}_3. \tag{1}$$

To find the equivalent representation of $\mathbf{w}$ in terms of the Doppler basis vectors, namely $\widetilde{\mathbf{w}}$, one simply takes the scalar product of the former with each member $\{\hat{\mathbf{r}}_i\}$ to yield

$$\widetilde{w}_i = \hat{\mathbf{r}}_i.\mathbf{w} = \sum_{j \in \mathcal{I}} r_{ij} w_j \ \text{ for all } i \in \mathcal{I}, \tag{2}$$

where $\hat{\mathbf{r}}_i = (r_{i1}, r_{i2}, r_{i3})^T$, for $i \in \mathcal{I}$, and thus $\widetilde{\mathbf{w}} = (\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3)^T$.

For each Doppler unit vector $\hat{\mathbf{r}}_i$, $i \in \mathcal{I}$, we may characterize its direction by:
$\theta_i \in [0, 2\pi)$, the azimuthal angle measured anti-clockwise from the x-axis; $\phi_i \in [-\pi/2, \pi/2]$, the elevation angle from the (x, y)-plane, taken to be positive when the z-coordinate is positive (as would normally be the case for a platform set on the ground measuring a point that is above ground). Each of the Doppler unit vectors can therefore be re-expressed in terms of the standard basis as follows:

$$\hat{\mathbf{r}}_i = \cos(\theta_i)\cos(\phi_i)\mathbf{e}_1 + \sin(\theta_i)\cos(\phi_i)\mathbf{e}_2 + \sin(\phi_i)\mathbf{e}_3 \quad i \in \mathcal{I}. \tag{3}$$

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From the expressions of (2) and (3), one can move from Cartesian coordinates to the equivalent Doppler coordinates via the following matrix equation:

$$\widetilde{\mathbf{w}} = \mathbf{M}\mathbf{w} \tag{4}$$

where

$$\mathbf{M}_{i1} = r_{i1} = \cos(\theta_i)\cos(\phi_i), \quad i \in \mathcal{I} \tag{5}$$

$$\mathbf{M}_{i2} = r_{i2} = \sin(\theta_i)\cos(\phi_i), \quad i \in \mathcal{I} \tag{6}$$

$$\mathbf{M}_{i3} = r_{i3} = \sin(\phi_i), \quad i \in \mathcal{I} \tag{7}$$

and

$$\widetilde{w}_i = \sum_{j\in\mathcal{I}} \mathbf{M}_{ij}w_j = \cos(\theta_i)\cos(\phi_i)w_1 + \sin(\theta_i)\cos(\phi_i)w_2 + \sin(\phi_i)w_3, \quad i \in \mathcal{I} \tag{8}$$

where it will be assumed that $\mathbf{M}$ is of full rank.

Conversely, given the 6 Doppler angles and the 3 Doppler wind velocity coordinates, one can map to the equivalent Cartesian coordinates via the following equation:

$$\mathbf{w} = \mathbf{M}^{-1}\widetilde{\mathbf{w}} \tag{9}$$

which, componentwise, is given by

$$w_j = \sum_{k\in\mathcal{I}} [\mathbf{M}^{-1}]_{jk}\widetilde{w}_k \quad j \in \mathcal{I}. \tag{10}$$

Generically, we set

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T \quad \boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T$$

for the azimuthal angles and elevation angles, respectively.

The true (Cartesian) velocity vector (at a given measurement position) will be denoted by $\mathbf{v}^{\text{true}} = (v_1^{\text{true}}, v_2^{\text{true}}, v_3^{\text{true}})^T$. Using the coordinate transformation equation (4), the Doppler velocity coordinate vector that *corresponds* to $\mathbf{v}^{\text{true}}$ when the angle orientations of the LIDARs, $(\boldsymbol{\theta}, \boldsymbol{\phi})$, are equal to $(\boldsymbol{\theta}^{\text{true}}, \boldsymbol{\phi}^{\text{true}})$, namely

$$\widetilde{\mathbf{v}}^{\text{c}}(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}^{\text{true}}, \boldsymbol{\phi}^{\text{true}}) = \left(\widetilde{v}_1^{\text{c}}(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}^{\text{true}}, \boldsymbol{\phi}^{\text{true}}), \widetilde{v}_2^{\text{c}}(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}^{\text{true}}, \boldsymbol{\phi}^{\text{true}}), \widetilde{v}_3^{\text{c}}(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}^{\text{true}}, \boldsymbol{\phi}^{\text{true}})\right)^T,$$

is equal to

$$\mathbf{M}(\boldsymbol{\theta}^{\text{true}}, \boldsymbol{\phi}^{\text{true}})\mathbf{v}^{\text{true}}. \tag{11}$$

The above argument motivates the following definition for reconstructing the (estimated) Cartesian wind velocity, given some arbitrary angle and Doppler velocity settings.

Definition 1 (Reconstruction Mapping)**.**
Suppose that each of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, is a 3×1 vector such that

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in [0, 2\pi)^3 \times [-\pi/2, \pi/2]^3 \times (-\infty, \infty)^3.$$

Define

$$\mathbf{v}(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \mathbf{M}^{-1}(\mathbf{a}, \mathbf{b})\mathbf{c}. \tag{12}$$

As hinted at just prior to the above definition, $\mathbf{v}(\boldsymbol{\theta}^{\text{true}}, \boldsymbol{\phi}^{\text{true}}, \widetilde{\mathbf{v}}^{\text{c}}(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}^{\text{true}}, \boldsymbol{\phi}^{\text{true}})) = \mathbf{v}^{\text{true}}$.

In the context of a particular measurement scenario, let $\boldsymbol{\theta}_0$ and $\boldsymbol{\phi}_0$ represent the demanded azimuthal angles and demanded elevation angles, respectively, and $\widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$ the Doppler velocities that had been recorded on the premise that the angles specifying the LIDAR orientations were equal to $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$, and that the true wind velocity is given by $\mathbf{v}^{\text{true}}$.

We represent the uncertainty around $\boldsymbol{\theta}_0$, $\boldsymbol{\phi}_0$, and $\widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$ by $\boldsymbol{\delta\theta}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))$, $\boldsymbol{\delta\phi}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))$ and $\boldsymbol{\delta\widetilde{v}}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))$, respectively; and set

$$\boldsymbol{\delta x}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))^T$$

$$:= \left[\boldsymbol{\delta\theta}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))^T, \boldsymbol{\delta\phi}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))^T, \boldsymbol{\delta\widetilde{v}}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))^T \right].$$

The uncertainties are modelled as random variables, parameterized by $\boldsymbol{\theta}_0$, $\boldsymbol{\phi}_0$, and $\widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$. The random variables that model the true azimuthal angles, true elevation angles, and true Doppler velocities, are given by

$$\boldsymbol{\theta}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)) := \boldsymbol{\theta}_0 + \boldsymbol{\delta\theta}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)); \tag{13}$$

$$\boldsymbol{\phi}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)) := \boldsymbol{\phi}_0 + \boldsymbol{\delta\phi}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)); \tag{14}$$

$$\widetilde{\mathbf{v}}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)) := \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0) + \boldsymbol{\delta\widetilde{v}}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)). \tag{15}$$

The assumptions that will be employed throughout the remainder of the analysis which, in essence, encapsulate the physics of the problem, are presented below.

Assumption 1 (LIDAR beam convergence within a locale of the measurement position).
The LIDAR beams will be assumed to be converging at the originally intended measurement position, $(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{r}_1, \widetilde{r}_2, \widetilde{r}_3)$ say, for some sufficiently small region $\boldsymbol{\epsilon}_{\text{locale}} \subseteq \mathbb{R}^6 \times \mathbb{R}^3$ such that $(\boldsymbol{\theta}, \boldsymbol{\phi}, \widetilde{r}_1, \widetilde{r}_2, \widetilde{r}_3) \in \boldsymbol{\epsilon}_{\text{locale}}$.

Assumption 2.
The components of $\boldsymbol{\delta x}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))$ are statistically independent of each other.

We will also be working extensively with the matrix $\mathbf{M}$, and its determinant, Δ, throughout the remainder of this paper.

Lemma 1 (determinant of the matrix $\mathbf{M}$).
The determinant of the matrix $\mathbf{M}$, denoted by $\Delta = \Delta(\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3)$, is given by

$$\Delta = \cos(\theta_1) \cos(\phi_1) \{\sin(\theta_2) \cos(\phi_2) \sin(\phi_3) - \sin(\phi_2) \sin(\theta_3) \cos(\phi_3)\}$$

$$- \sin(\theta_1) \cos(\phi_1) \{\sin(\phi_3) \cos(\theta_2) \cos(\phi_2) - \sin(\phi_2) \cos(\theta_3) \cos(\phi_3)\}$$

$$+ \sin(\phi_1) \{\cos(\theta_2) \cos(\phi_2) \sin(\theta_3) \cos(\phi_3) - \sin(\theta_2) \cos(\phi_2) \cos(\theta_3) \cos(\phi_3)\}. \tag{16}$$

or, equivalently,

$$\Delta = \sum_{\pi \in S_3} \text{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) \tag{17}$$

where S_3 is the set of all permutations on $\mathcal{I}$.

Proof

The results follow trivially by applying a cofactor expansion along the first row of $\mathbf{M}$ in order to obtain (16), and the Leibniz formula to $\mathbf{M}$ for (17). □

As, and when, it proves to be convenient, we will forgo mention of $(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$ from $\widetilde{\mathbf{v}}_0$, for the sake of notational simplicity. We will adopt a similar strategy for $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ in relation to $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)$.

3 Distributional results for the reconstructed velocity vector

3.1 No uncertainty in the demanded angles but uniform uncertainty in the Doppler velocity measurements

We first consider the case in which the LIDAR orientation angles can be selected to take particular values without any uncertainty involved i.e. that $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)$ can be taken to be equal to $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$, which is encapsulated within the following assumption set that follows:

Assumption 3. *For a given, nominal, parameter setting $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, the range for the Doppler velocity $\widetilde{\mathbf{v}}^*$ is given by*

$$R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \left\{ \mathbf{z} : \widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \leq \mathbf{z} \leq \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}$$

where

$$\mathbf{z} = (z_1, z_2, z_3)^T;$$
$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \left(\widetilde{v}_1^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \widetilde{v}_2^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \widetilde{v}_3^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)^T$$
$$\widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \left(\widetilde{v}_1^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \widetilde{v}_2^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \widetilde{v}_3^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)^T$$
$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) < \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0).$$

It is also assumed that there is no uncertainty in the values of $\boldsymbol{\theta}^*$ and $\boldsymbol{\phi}^*$ such that $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*) = (\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$.

Denote the joint probability density function of $\widetilde{\mathbf{v}}^*$ by $f_{\widetilde{\mathbf{v}}^*}(\cdot)$, with aforementioned range $R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$.

The reconstructed velocity vector, parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, is given by

$$\mathbf{v}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \mathbf{v}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))$$
$$= \mathbf{M}^{-1}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\widetilde{\mathbf{v}}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0). \tag{18}$$

Definition 2 (Range of $\mathbf{v}^*$).
Under Assumption 3, the range of $\mathbf{v}^*$, parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, is given by

$$R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \left\{ \mathbf{w}' : \exists \, \widetilde{\mathbf{v}}' \in R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \text{ s.t. } \mathbf{w}' = \mathbf{M}^{-1}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\widetilde{\mathbf{v}}' \right\}. \tag{19}$$

It should be clear from the above definition that $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ encapsulates the set of points for $\mathbf{v}^*$ that can be reached, under the parameter configuration $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, albeit the possibility that some members of that set may have zero density is not precluded.

Denote the joint probability density function of $\mathbf{v}^*$ by $f_{\mathbf{v}^*}(\cdot)$, with associated range $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$.

We will also work with the notion of Lebesgue measure and Lebesgue measurable sets (see [2] or [16], for e.g.).

Theorem 1.

Under Assumption 3, the following hold true:

(i)

$$R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \equiv \left\{ \mathbf{w}' : \widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \leq \mathbf{M}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\mathbf{w}' \leq \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\};$$

(ii)

$$\mathrm{vol}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) = \frac{\prod_{i=1}^3 \left(\widetilde{v}_i^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) - \widetilde{v}_i^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)}{|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|} = \frac{\mathrm{vol}(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))}{|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|}$$

where $\mathrm{vol}(\cdot)$ *denotes Lebesgue measure on* $\mathbb{R}^3$ *(thus representing volume in the "standard" way).*

Proof

Define

$$R_{\mathbf{v}^*}^{\mathrm{candidate}}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \left\{ \mathbf{w}' : \widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \leq \mathbf{M}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\mathbf{w}' \leq \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}. \tag{20}$$

Suppose $\mathbf{v}' \in R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. By the definition of $\mathbf{v}^*$, and Assumption 3, there must exist a $\widetilde{\mathbf{v}}'$ satisfying

$$\mathbf{v}' = \mathbf{M}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)^{-1}\widetilde{\mathbf{v}}' \tag{21}$$

such that $\widetilde{\mathbf{v}}' \in R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, i.e.

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \leq \widetilde{\mathbf{v}}' \leq \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0). \tag{22}$$

However, by (21) and (22), it follows that

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \leq \mathbf{M}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\mathbf{v}' \leq \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \tag{23}$$

as required to show that $\mathbf{v}' \in R_{\mathbf{v}^*}^{\mathrm{candidate}}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, and hence that

$$R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \subseteq R_{\mathbf{v}^*}^{\mathrm{candidate}}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0).$$

Conversely, suppose that $\mathbf{v}' \in R_{\mathbf{v}^*}^{\mathrm{candidate}}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$: then (23) holds. Setting $\widetilde{\mathbf{v}}^{\mathrm{new}} := \mathbf{M}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\mathbf{v}'$, it follows that $\widetilde{\mathbf{v}}^{\mathrm{new}} \in R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. However, by the definition of $\widetilde{\mathbf{v}}^{\mathrm{new}}$, it is also the case that $\mathbf{v}' = \mathbf{M}^{-1}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\widetilde{\mathbf{v}}^{\mathrm{new}}$ which, on account of the definition of (18), implies that $\mathbf{v}' \in R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, and hence that $R_{\mathbf{v}^*}^{\mathrm{candidate}}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \subseteq R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. Thus (i) is established.

To establish (ii), whilst suppressing display of $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ for the sake of notational brevity, first note that $R_{\widetilde{\mathbf{v}}^*}$ can be expressed as

$$R_{\widetilde{\mathbf{v}}^*} = \big\{ \widetilde{\mathbf{z}} \in \mathbb{R}^3 : \widetilde{\mathbf{z}} = (\widetilde{v}_1^{\min}, \widetilde{v}_2^{\min}, \widetilde{v}_3^{\min})^T$$

$$+ \lambda_1(\widetilde{v}_1^{\max} - \widetilde{v}_1^{\min})\mathbf{e}_1 + \lambda_2(\widetilde{v}_2^{\max} - \widetilde{v}_2^{\min})\mathbf{e}_2 + \lambda_3(\widetilde{v}_3^{\max} - \widetilde{v}_3^{\min})\mathbf{e}_3; \ \lambda_i \in [0,1], i \in \mathcal{I} \big\}.$$

However, by (18),

$$R_{\mathbf{v}^*} = \left\{ \mathbf{z} : \mathbf{z} = \mathbf{M}^{-1}\widetilde{\mathbf{z}}; \ \widetilde{\mathbf{z}} \in R_{\widetilde{\mathbf{v}}^*} \right\}.$$

Defining

$$\mathbf{H} := \mathbf{M}^{-1}\mathrm{diag}(\widetilde{v}_1^{\max} - \widetilde{v}_1^{\min}, \widetilde{v}_2^{\max} - \widetilde{v}_2^{\min}, \widetilde{v}_3^{\max} - \widetilde{v}_3^{\min}),$$

$$\mathbf{h}_0 := \mathbf{M}^{-1} \begin{bmatrix} \widetilde{v}_1^{\min} \\ \widetilde{v}_2^{\min} \\ \widetilde{v}_3^{\min} \end{bmatrix},$$

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and
$$\mathbf{h}_1 = \mathbf{H}\mathbf{e}_1, \quad \mathbf{h}_2 = \mathbf{H}\mathbf{e}_2, \quad \mathbf{h}_3 = \mathbf{H}\mathbf{e}_3,$$

then it follows that
$$R_{\mathbf{v}^*} = \big\{ \mathbf{z} : \mathbf{z} = \mathbf{M}^{-1}(\widetilde{v}_1^{\min}, \widetilde{v}_2^{\min}, \widetilde{v}_3^{\min})^T$$
$$+ \lambda_1(\widetilde{v}_1^{\max} - \widetilde{v}_1^{\min})\mathbf{M}^{-1}\mathbf{e}_1 + \lambda_2(\widetilde{v}_2^{\max} - \widetilde{v}_2^{\min})\mathbf{M}^{-1}\mathbf{e}_2 + \lambda_3(\widetilde{v}_3^{\max} - \widetilde{v}_3^{\min})\mathbf{M}^{-1}\mathbf{e}_3; \ \lambda_i \in [0,1], i \in \mathcal{I} \big\}$$
$$= \big\{ \mathbf{z} : \mathbf{z} = \mathbf{h}_0 + \lambda_1\mathbf{h}_1 + \lambda_2\mathbf{h}_2 + \lambda_3\mathbf{h}_3; \ \lambda_i \in [0,1], i \in \mathcal{I} \big\}$$

which, on account of $\mathbf{h}_1$, $\mathbf{h}_2$ and $\mathbf{h}_3$ being linearly independent (recall that $\mathbf{M}$ is assumed to be of full rank), delineates a parallelepiped.

Relative to $\mathbf{h}_0$, the position vectors of the vertices that are directly adjacent to $\mathbf{h}_0$, corresponding to $(\lambda_1, \lambda_2, \lambda_3) \in \{0,1\}^3$ such that $\sum_{i=1}^3 \lambda_i = 1$, are given by $\mathbf{h}_1$, $\mathbf{h}_2$, and $\mathbf{h}_3$. Since $\mathbf{H} = [\mathbf{h}_1 | \mathbf{h}_2 | \mathbf{h}_3]$, then it follows (from, for example, Theorem 3.5.4 (b) of [1], or Section 7.6.3 of [15]) that

$$\mathrm{vol}(R_{\mathbf{v}^*}) = |\det(\mathbf{H})| = |\det(\mathbf{M}^{-1})||\det(\mathrm{diag}(\widetilde{v}_1^{\max} - \widetilde{v}_1^{\min}, \widetilde{v}_2^{\max} - \widetilde{v}_2^{\min}, \widetilde{v}_3^{\max} - \widetilde{v}_3^{\min}))|$$

$$= \frac{1}{|\det(\mathbf{M})|} \prod_{i=1}^3 (\widetilde{v}_i^{\max} - \widetilde{v}_i^{\min}) = \frac{1}{|\Delta|} \prod_{i=1}^3 (\widetilde{v}_i^{\max} - \widetilde{v}_i^{\min}).$$

$\square$

Remark

It is perhaps worth noting that $0 < |\Delta| \le 1$, and that $1/|\Delta|$ has the effect of multiplying the volume of $R_{\widetilde{\mathbf{v}}^*}$ by a scale factor larger than 1 in order to yield the volume of $R_{\mathbf{v}^*}$ (and hence acting as an inflation factor that is applied to the volume of the Doppler velocity uncertainty range in order to yield that of the reconstructed velocity uncertainty range), except when $|\Delta| = 1$ (in which case the scale factor is unity whence there is no inflation). Where indeed the value of $|\Delta|$ happens to fall within the range $(0, 1]$ will depend on the configuration of the LIDAR beams.

Corollary 1. *Under Assumption 3, suppose that the components of $\widetilde{\mathbf{v}}^*$ are uniform and mutually independent. Then*
$$f_{\mathbf{v}^*}(\mathbf{v}') = \frac{|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|}{\prod_{i=1}^3 \left(\widetilde{v}_i^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) - \widetilde{v}_i^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right)}$$
for $\mathbf{v}' \in R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$.

Proof

Given that the coordinate sets $R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ and $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ can both be delineated within the same vector space in $\mathbb{R}^3$, then the result follows from standard properties regarding the application of a linear transformation to a Lebesgue measurable set, so that for $\mathbf{v}' \in R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$,
$f_{\mathbf{v}^*}(\mathbf{v}') = \frac{1}{\mathrm{vol}(R_{\mathbf{v}^*})}.$ $\square$

3.2 Uniform uncertainty in both the demanded angles and Doppler velocity measurements

We now generalize to the case in which all three of $\boldsymbol{\theta}^*$, $\boldsymbol{\phi}^*$ and $\widetilde{\mathbf{v}}^*$ are considered to be uncertain.

Assumption 4. *For a given, nominal, parameter setting $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, the ranges of $\boldsymbol{\theta}^*$, $\boldsymbol{\phi}^*$, and $\widetilde{\mathbf{v}}^*$, are given by*
$$R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \big\{ \mathbf{x} : \boldsymbol{\theta}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \mathbf{x} \le \boldsymbol{\theta}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \big\}$$

$$R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \left\{ \mathbf{y} : \boldsymbol{\phi}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \mathbf{y} \le \boldsymbol{\phi}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}$$

$$R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \left\{ \mathbf{z} : \widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \mathbf{z} \le \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}$$

where

$$\mathbf{x} = (x_1, x_2, x_3)^T, \ \mathbf{y} = (y_1, y_2, y_3)^T, \ \mathbf{z} = (z_1, z_2, z_3)^T;$$

$$\boldsymbol{\theta}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) = \left(\theta_1^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \theta_2^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \theta_3^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)^T$$

$$\boldsymbol{\theta}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) = \left(\theta_1^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \theta_2^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \theta_3^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)^T$$

$$\boldsymbol{\theta}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) < \boldsymbol{\theta}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0);$$

$$\boldsymbol{\phi}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) = \left(\phi_1^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \phi_2^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \phi_3^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)^T$$

$$\boldsymbol{\phi}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) = \left(\phi_1^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \phi_2^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \phi_3^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)^T$$

$$\boldsymbol{\phi}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) < \boldsymbol{\phi}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0);$$

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) = \left(\widetilde{v}_1^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \widetilde{v}_2^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \widetilde{v}_3^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)^T$$

$$\widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) = \left(\widetilde{v}_1^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \widetilde{v}_2^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \widetilde{v}_3^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)^T$$

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) < \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0).$$

Further,
the range of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)$ is given by

$$R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$$

and the range of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)$ is given by

$$R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0).$$

In this section, we define
$$\mathbf{v}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))$$

$$:= \mathbf{v}(\boldsymbol{\theta}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)), \boldsymbol{\phi}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)), \widetilde{\mathbf{v}}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)))$$

$$= \mathbf{M}^{-1}(\boldsymbol{\theta}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)), \boldsymbol{\phi}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)))\widetilde{\mathbf{v}}^*(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0)) \quad (24)$$

and thus $\mathbf{v}^*$ is parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0(\mathbf{v}^{\text{true}}; \boldsymbol{\theta}_0, \boldsymbol{\phi}_0))$; and denote the joint probability density function of $\mathbf{v}^*$ by $f_{\mathbf{v}^*}(\cdot)$, with associated range $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$.

In the next few definitions and following lemma, we describe the joint range of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)$, along with associated quantities.

Definition 3 (Ranges).
Under Assumption 4:
(i) the range of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)$, parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, is given by

$$R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \Big\{ (\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') : \ (\boldsymbol{\theta}', \boldsymbol{\phi}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0);$$

$$\exists \, \widetilde{\mathbf{v}}' \in R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \ s.t. \ \mathbf{v}' = \mathbf{M}^{-1}(\boldsymbol{\theta}', \boldsymbol{\phi}')\widetilde{\mathbf{v}}' \Big\}; \quad (25)$$

(ii) the range of $\mathbf{v}^*$, parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, is given by

$$R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \bigcup_{(\mathbf{x}', \mathbf{y}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} \left\{ \mathbf{w}' : (\mathbf{x}', \mathbf{y}', \mathbf{w}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\} ; \qquad (26)$$

(iii) the range of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)$ conditional on $\mathbf{v}^* = \mathbf{v}'$, parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, is given by

$$R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)|\mathbf{v}^* = \mathbf{v}'}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \left\{ (\mathbf{x}', \mathbf{y}') : (\mathbf{x}', \mathbf{y}', \mathbf{v}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\} . \qquad (27)$$

It should be clear that each of the quantities defined in $(i) - (iii)$ above do indeed represent the set of points that can be reached by $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)$, $\mathbf{v}^*$, and $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)$ when $\mathbf{v}^*$ is constrained to take the value $\mathbf{v}'$, respectively. Also, the possibility that there may exist points within any of those sets whose values for the associated probability density function is zero is not precluded.

Next we consider the likely possible more explicit expression for $R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$.
Let

$$\mathcal{R}^{\text{candidate}}_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \Big\{ (\mathbf{x}', \mathbf{y}', \mathbf{w}') : \mathbf{x}' \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0); \ \mathbf{y}' \in R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0);$$

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \mathbf{M}(\mathbf{x}', \mathbf{y}')\mathbf{w}' \le \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \Big\} \qquad (28)$$

and define the mapping $\mathbf{P}$ such that

$$\mathbf{P} : \mathcal{R}^{\text{candidate}}_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \mapsto R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$$

and represented by the matrix equation $\widetilde{\mathbf{v}}' = \mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}')\mathbf{v}'$, for $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}')$ in the domain and $\widetilde{\mathbf{v}}'$ in the co-domain.

Thus $\mathbf{P}$ takes in a realization of the 9 variables held within $\boldsymbol{\theta}^*$, $\boldsymbol{\phi}^*$ and $\mathbf{v}^*$ to yield a realization of the 3 variables held within $\widetilde{\mathbf{v}}^*$. In order to (eventually) find the joint probability density function of $\mathbf{v}^*$, we need to augment the co-domain of $\mathbf{P}$ with that which is associated with suitable additional variables, and adjust the matrix equation that maps from the domain to the new co-domain accordingly. To that end, define the mapping $\mathbf{Q}$, such that

$$\mathbf{Q} : \mathcal{R}^{\text{candidate}}_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \mapsto R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$$

represented by the matrix equation

$$\left[\begin{array}{c} \boldsymbol{\theta}' \\ \hline \boldsymbol{\phi}' \\ \hline \widetilde{\mathbf{v}}' \end{array} \right] = \left[\begin{array}{c|c} \text{diag}(1, 1, 1, 1, 1, 1) & \mathbf{0}_{6 \times 3} \\ \hline \mathbf{0}_{3 \times 6} & \mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}') \end{array} \right] \left[\begin{array}{c} \boldsymbol{\theta}' \\ \hline \boldsymbol{\phi}' \\ \hline \mathbf{v}' \end{array} \right] \qquad (29)$$

for $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}')$ in the domain and $(\boldsymbol{\theta}', \boldsymbol{\phi}', \widetilde{\mathbf{v}}')$ in the co-domain.

Lemma 2.
Under Assumption 4,
(i) the range of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)$, denoted by $R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, is such that

$$R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \equiv \mathcal{R}^{\text{candidate}}_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$$

$$= \left\{ (\mathbf{x}', \mathbf{y}', \mathbf{w}') : \mathbf{x}' \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0); \ \mathbf{y}' \in R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0); \right.$$

$$\left. \widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \mathbf{M}(\mathbf{x}', \mathbf{y}')\mathbf{w}' \le \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}; \tag{30}$$

(ii) the range of $\mathbf{v}^*$, denoted by $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, is such that

$$R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \equiv \bigcup_{(\mathbf{x}', \mathbf{y}') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} \left\{ \mathbf{w}' : \widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \mathbf{M}(\mathbf{x}', \mathbf{y}')\mathbf{w}' \le \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}; \tag{31}$$

(iii) the range of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)$ conditional on $\mathbf{v}^* = \mathbf{v}'$, denoted by $R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)|\mathbf{v}^*=\mathbf{v}'}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, is such that

$$R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)|\mathbf{v}^*=\mathbf{v}'}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \equiv \left\{ (\mathbf{x}', \mathbf{y}') : \mathbf{x}' \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0); \ \mathbf{y}' \in R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0); \right.$$

$$\left. \widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \mathbf{M}(\mathbf{x}', \mathbf{y}')\mathbf{v}' \le \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}. \tag{32}$$

Proof

Suppose $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. Then by the definition of $R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, there must exist a $\widetilde{\mathbf{v}}'$ satisfying

$$\mathbf{v}' = \mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}')^{-1}\widetilde{\mathbf{v}}' \tag{33}$$

such that $(\boldsymbol{\theta}', \boldsymbol{\phi}', \widetilde{\mathbf{v}}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. By Assumption 4, it therefore follows that

$$\boldsymbol{\theta}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \boldsymbol{\theta}' \le \boldsymbol{\theta}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \tag{34}$$

$$\boldsymbol{\phi}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \boldsymbol{\phi}' \le \boldsymbol{\phi}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \tag{35}$$

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \widetilde{\mathbf{v}}' \le \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0). \tag{36}$$

However, (33) and (36) imply that

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}')\mathbf{v}' \le \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0). \tag{37}$$

The fact that $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}')$ satisfies (34), (35), and (37), suffices to show that $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \in \mathcal{R}^{\text{candidate}}_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. Hence $R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \subseteq \mathcal{R}^{\text{candidate}}_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$.
Conversely, suppose that $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \in \mathcal{R}^{\text{candidate}}_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. Then

$$\boldsymbol{\theta}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \boldsymbol{\theta}' \le \boldsymbol{\theta}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$$

$$\boldsymbol{\phi}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \boldsymbol{\phi}' \le \boldsymbol{\phi}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$$

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \le \mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}')\mathbf{v}' \le \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0).$$

Setting $\widetilde{\mathbf{v}}^{\text{new}} := \mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}')\mathbf{v}'$, it follows that $(\boldsymbol{\theta}', \boldsymbol{\phi}', \widetilde{\mathbf{v}}^{\text{new}}) \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. Additionally, by the definition of $\widetilde{\mathbf{v}}^{\text{new}}$, it also follows that $\mathbf{v}' = \mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}')^{-1}\widetilde{\mathbf{v}}^{\text{new}}$, and hence

$$(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0).$$

Hence $\mathcal{R}^{\text{candidate}}_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \subseteq R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. The result of (i) now follows. Part (ii) follows from combining (26) with (30). Part (iii) follows from combining (27) with (30). $\square$

We are ultimately interested in bounding the volume occupied by the set of points within which $\mathbf{v}^*$ could conceivably reside: this will be facilitated by our being able to access an expression for the probability density function of $\mathbf{v}^*$, which we shall extract from the joint probability density function of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)$.

Proposition 1.
Under Assumption 4, suppose that all 9 components of $\boldsymbol{\theta}^*$, $\boldsymbol{\phi}^*$ and $\widetilde{\mathbf{v}}^*$ are jointly uniform and independent.

For $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$,

$$f_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') = \frac{\left| \Delta(\boldsymbol{\theta}', \boldsymbol{\phi}') \right|}{\left[\prod_{i=1}^{3} L_{\theta_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right] \left[\prod_{i=1}^{3} L_{\phi_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right] \left[\prod_{i=1}^{3} L_{\widetilde{v}_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right]}$$

where

$$L_{\theta_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \theta_i^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) - \theta_i^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \ i \in \mathcal{I}$$

$$L_{\phi_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \phi_i^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) - \phi_i^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \ i \in \mathcal{I}$$

$$L_{\widetilde{v}_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \widetilde{v}_i^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) - \widetilde{v}_i^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0), \ i \in \mathcal{I}.$$

Proof
Suppose that $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)$ takes the value $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$: then it follows that there exists a unique (on account of $\mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}')$ being assumed to be of full rank) $\widetilde{\mathbf{v}}'$ such that $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)$ takes the value $(\boldsymbol{\theta}', \boldsymbol{\phi}', \widetilde{\mathbf{v}}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)}$ for which $\mathbf{v}' = \mathbf{M}^{-1}(\boldsymbol{\theta}', \boldsymbol{\phi}')\widetilde{\mathbf{v}}'$.
Define

$$\nabla_{(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}')} = \left(\frac{\partial}{\partial \theta_1'}, \frac{\partial}{\partial \theta_2'}, \frac{\partial}{\partial \theta_3'}, \frac{\partial}{\partial \phi_1'}, \frac{\partial}{\partial \phi_2'}, \frac{\partial}{\partial \phi_3'}, \frac{\partial}{\partial v_1'}, \frac{\partial}{\partial v_2'}, \frac{\partial}{\partial v_3'} \right)^T.$$

From standard distribution theory, it can be deduced that, for $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}$,

$$f_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') = f_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)}(\boldsymbol{\theta}', \boldsymbol{\phi}', \widetilde{\mathbf{v}}'(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}'))|J_{(\boldsymbol{\theta}', \boldsymbol{\phi}', \widetilde{\mathbf{v}}'(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}'))}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}')|$$

where, the Jacobian determinant, $J_{(\boldsymbol{\theta}', \boldsymbol{\phi}', \widetilde{\mathbf{v}}'(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}'))}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') = \det \mathbf{G}$, such that $\mathbf{G} = \mathbf{D}^T$ and

$$\mathbf{D} = \nabla_{(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}')}[(\boldsymbol{\theta}')^T|(\boldsymbol{\phi}')^T|(\widetilde{\mathbf{v}}'(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}'))^T]$$

(c.f. Section 2.7 of [5]).
It follows that

$$\mathbf{G} = \left[\begin{array}{c|c|c} \text{diag}(1,1,1) & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \hline \mathbf{0}_{3\times 3} & \text{diag}(1,1,1) & \mathbf{0}_{3\times 3} \\ \hline \mathbf{F}_{\boldsymbol{\theta}'}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') & \mathbf{F}_{\boldsymbol{\phi}'}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') & \mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}') \end{array} \right]$$

where, for $i, j \in \mathcal{I}$,

$$\left[\mathbf{F}_{\boldsymbol{\theta}'}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \right]_{ij} = \sum_{k=1}^{3} v_k' \frac{\partial [\mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}')]_{ik}}{\partial \theta_j'}$$

and

$$\left[\mathbf{F}_{\boldsymbol{\phi}'}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \right]_{ij} = \sum_{k=1}^{3} v_k' \frac{\partial [\mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}')]_{ik}}{\partial \phi_j'}.$$

By using a co-factor expansion along the first row of $\mathbf{G}$, it is easy to see (notwithstanding the composition of $\mathbf{F}_{\boldsymbol{\theta}'}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}')$ and $\mathbf{F}_{\boldsymbol{\phi}'}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}')$, other than to note that their components are well-defined and finite) that

$$\det \mathbf{G} = \det \mathbf{M}.$$

Thus, by independence of the 9 variables in $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}^*)$, one obtains

$$f_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') = |\Delta(\boldsymbol{\theta}', \boldsymbol{\phi}')|$$

$$\times \left[\prod_{i=1}^{3} \frac{\mathbf{1}_{[\theta_i^{\min}, \theta_i^{\max}]}(\theta_i')}{L_{\theta_i}} \right] \left[\prod_{i=1}^{3} \frac{\mathbf{1}_{[\phi_i^{\min}, \phi_i^{\max}]}(\phi_i')}{L_{\phi_i}} \right] \left[\prod_{i=1}^{3} \frac{\mathbf{1}_{[\widetilde{v}_i^{\min}, \widetilde{v}_i^{\max}]}\left(\sum_{j=1}^{3} \mathbf{M}_{ij}(\boldsymbol{\theta}', \boldsymbol{\phi}')v_j' \right)}{L_{\widetilde{v}_i}} \right]$$

where $\mathbf{1}_A(\cdot)$ is the indicator function on the set A. Therefore

$$f_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') = \frac{|\Delta(\boldsymbol{\theta}', \boldsymbol{\phi}')|}{\left[\prod_{i=1}^{3} L_{\theta_i} \right] \left[\prod_{i=1}^{3} L_{\phi_i} \right] \left[\prod_{i=1}^{3} L_{\widetilde{v}_i} \right]}$$

since, by assumption, $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, and by recalling Lemma 2. $\qquad \square$

Lemma 3.
Under Assumption 4, and the parameter setting $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, suppose that all 9 components of $\boldsymbol{\theta}^*$, $\boldsymbol{\phi}^*$ and $\widetilde{\mathbf{v}}^*$ are jointly uniform and independent. Then

$$f_{\mathbf{v}^*}(\mathbf{v}') = \frac{1}{\left[\prod_{i=1}^{3} L_{\theta_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right] \left[\prod_{i=1}^{3} L_{\phi_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right] \left[\prod_{i=1}^{3} L_{\widetilde{v}_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right]}$$

$$\times \int_{R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)|\mathbf{v}^*=\mathbf{v}'}} |\Delta(\boldsymbol{\theta}', \boldsymbol{\phi}')| \; d(\boldsymbol{\theta}', \boldsymbol{\phi}') \tag{38}$$

for $\mathbf{v}' \in R_{\mathbf{v}^*}$.

Proof
This is just an exercise in using standard results for extracting the marginal p.d.f. of $\mathbf{v}^*$, evaluated at $\mathbf{v}'$, from the joint p.d.f of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)$ when $\mathbf{v}^* = \mathbf{v}'$: this is done by integrating out the joint p.d.f. of $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)$, when $\mathbf{v}^* = \mathbf{v}'$, with respect to those values taken by $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)$, $(\boldsymbol{\theta}', \boldsymbol{\phi}')$ say, for which $(\boldsymbol{\theta}', \boldsymbol{\phi}', \mathbf{v}') \in R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \mathbf{v}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. $\qquad \square$

High dimensional integrals are commonplace in uncertainty quantification problems, of which their calculation is sometimes circumvented by working with an integrand defined on a lower dimensional space if such an approximation can be warranted: Wang [17] employed just such a technique in the calculation of certain statistical moments. In our context and problem we end up bounding integrals of the form that appear in (38) by maximizing/minimizing $|\Delta(\cdot, \cdot)|$ over an appropriate hyper-rectangular domain.

Theorem 2.
Under Assumption 4, suppose that all 9 components of $\boldsymbol{\theta}^*$, $\boldsymbol{\phi}^*$ and $\widetilde{\mathbf{v}}^*$ are jointly uniform and independent. Then

$$\mathrm{vol}\left(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right) \geq \underline{\mathrm{vol}}\left(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right) \tag{39}$$

where

$$\underline{\mathrm{vol}}\left(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right) := \frac{\mathrm{vol}\left(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right)}{\max\left\{ |\Delta(\boldsymbol{\theta}'', \boldsymbol{\phi}'')| : (\boldsymbol{\theta}'', \boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}}. \tag{40}$$

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Proof

From Lemma 2 (iii), one deduces that $R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)|\mathbf{v}^*=\mathbf{v}'}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \subseteq R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, and hence

$$\int_{R_{(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)|\mathbf{v}^*=\mathbf{v}'}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta}', \boldsymbol{\phi}')| \; d(\boldsymbol{\theta}', \boldsymbol{\phi}') \leq \int_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta}', \boldsymbol{\phi}')| \; d(\boldsymbol{\theta}', \boldsymbol{\phi}')$$

$$\leq \int_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} \max\left\{ |\Delta(\boldsymbol{\theta}'', \boldsymbol{\phi}'')| : (\boldsymbol{\theta}'', \boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\} \; d(\boldsymbol{\theta}', \boldsymbol{\phi}')$$

$$= \mathrm{vol}(R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) \mathrm{vol}(R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))$$

$$\times \quad \max\left\{ |\Delta(\boldsymbol{\theta}'', \boldsymbol{\phi}'')| : (\boldsymbol{\theta}'', \boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}.$$

However, noting that

$$\mathrm{vol}(R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) = \prod_{i=1}^{3} L_{\theta_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$$

and

$$\mathrm{vol}(R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) = \prod_{i=1}^{3} L_{\phi_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$$

then it now follows that

$$f_{\mathbf{v}^*}(\mathbf{v}') \leq \frac{\max\left\{ |\Delta(\boldsymbol{\theta}'', \boldsymbol{\phi}'')| : (\boldsymbol{\theta}'', \boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}}{\prod_{i=1}^{3} L_{\widetilde{v}_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)}$$

for $\mathbf{v}' \in R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$. On the other hand, since

$$\int_{R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} f_{\mathbf{v}^*}(\mathbf{v}') d\mathbf{v}' = 1$$

then

$$1 \leq \frac{\max\left\{ |\Delta(\boldsymbol{\theta}'', \boldsymbol{\phi}'')| : (\boldsymbol{\theta}'', \boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}}{\prod_{i=1}^{3} L_{\widetilde{v}_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} \mathrm{vol}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))$$

and on account of the fact that $\mathrm{vol}(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) = \prod_{i=1}^{3} L_{\widetilde{v}_i}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, then the result follows. $\square$

For our uncertainty quantification exercise, we shall additionally consider working with a slightly different metric which is readily amenable to finding both lower and upper bounds that are reasonably tight for the application scenarios that we have in mind.

Definition 4 (Conditional Range of $\mathbf{v}^*$ given $(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)$)**.**
Under Assumption 4, and $(\boldsymbol{\theta}', \boldsymbol{\phi}') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, *define*

$$R_{\mathbf{v}^*|(\boldsymbol{\theta}', \boldsymbol{\phi}')}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) := \left\{ \mathbf{w}' : \widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \leq \mathbf{M}(\boldsymbol{\theta}', \boldsymbol{\phi}') \mathbf{w}' \leq \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \right\}.$$

Definition 5 (Volume of conditional range of $\mathbf{v}^*$ "averaged" across the angle uncertainty range)**.**
Under Assumption 4, define

$$\text{vol}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) := E\left[\text{vol}\left(R_{\mathbf{v}^*|(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right)\right]. \tag{41}$$

Theorem 3.
Under Assumption 4, suppose that all 6 components of $\boldsymbol{\theta}^*$ and $\boldsymbol{\phi}^*$ are jointly uniform and independent. Then

$$\underline{\text{vol}}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) \leq \text{vol}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) \leq \overline{\text{vol}}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) \tag{42}$$

where

$$\underline{\text{vol}}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) := \frac{\text{vol}(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}_0))}{\max\left\{|\Delta(\boldsymbol{\theta}'', \boldsymbol{\phi}'')| : (\boldsymbol{\theta}'', \boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right\}} \tag{43}$$

and

$$\overline{\text{vol}}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) := \frac{\text{vol}(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*, \widetilde{\mathbf{v}}_0))}{\min\left\{|\Delta(\boldsymbol{\theta}'', \boldsymbol{\phi}'')| : (\boldsymbol{\theta}'', \boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right\}}. \tag{44}$$

Proof

$$\text{vol}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)) = E\left[\text{vol}\left(R_{\mathbf{v}^*|(\boldsymbol{\theta}^*, \boldsymbol{\phi}^*)}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right)\right]$$

$$= \int_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} \text{vol}\left(R_{\mathbf{v}^*|(\boldsymbol{\theta}', \boldsymbol{\phi}')}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right) \frac{d(\boldsymbol{\theta}', \boldsymbol{\phi}')}{\text{vol}(R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))\text{vol}(R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))}$$

$$= \int_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} \frac{\text{vol}(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))}{|\Delta(\boldsymbol{\theta}', \boldsymbol{\phi}')|} \frac{d(\boldsymbol{\theta}', \boldsymbol{\phi}')}{\text{vol}(R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))\text{vol}(R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))}$$

$$\left\{\begin{array}{l} \geq \dfrac{\text{vol}(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))}{\max\left\{|\Delta(\boldsymbol{\theta}'', \boldsymbol{\phi}'')| : (\boldsymbol{\theta}'', \boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right\}} \\[3mm] \leq \dfrac{\text{vol}(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))}{\min\left\{|\Delta(\boldsymbol{\theta}'', \boldsymbol{\phi}'')| : (\boldsymbol{\theta}'', \boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right\}} \end{array}\right\}$$

$$\times \int_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} \frac{d(\boldsymbol{\theta}', \boldsymbol{\phi}')}{\text{vol}(R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))\text{vol}(R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))}$$

where the third line follows using an analogous argument to the proof of Theorem 1 part (ii). Noting that the value of the integral in the final line is equal to 1, the claimed lower and upper bounds then follow. $\qquad\square$

4 Numerical optimization of $|\Delta(\cdot, \cdot)|$ across $R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$

In the absence of readily obtainable values for the global minimum and maximum of $|\Delta(\cdot, \cdot)|$ across $R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, one may wish to employ numerical optimization techniques in order to gauge their values. In particular, we seek to find the value of the global minimum/maximum over the search region to within an acceptable tolerance level. There is an extensive literature that addresses the topic of numerical techniques for global optimization ([3], [4], [13], [14]).

One could simply construct equally spaced grid points for $R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ and $R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, given by $\widehat{R}_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ and $\widehat{R}_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, respectively, and calculate the value of $|\Delta(\cdot, \cdot)|$ for each point within $\widehat{R}_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times \widehat{R}_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, in order to yield the largest/smallest of these values in the case of maximization/minimization. In using such a *grid search* approach, a determination needs to made as to what the widest permissible spacing between adjacent points within the grid ought to be in order to ensure that the optimum is correct to within a certain level of accuracy.

In view of the fact that $\Delta(\cdot, \cdot)$ is twice continuously differentiable, then we derive meaningful bounds on both its gradient and Hessian, in order to determine the appropriate spacing between the grid points. To this end, we first introduce the following lemma.

Lemma 4.
Define

$$\boldsymbol{\delta\theta}' := (\delta\theta_1', \delta\theta_2', \delta\theta_3')^T$$

$$\boldsymbol{\delta\phi}' := (\delta\phi_1', \delta\phi_2', \delta\phi_3')^T$$

$$\boldsymbol{\delta y} := (\boldsymbol{\delta\theta}'^T, \boldsymbol{\delta\phi}'^T)^T$$

$$\nabla_{(\boldsymbol{\theta},\boldsymbol{\phi})} := \left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \frac{\partial}{\partial\theta_3}, \frac{\partial}{\partial\phi_1}, \frac{\partial}{\partial\phi_2}, \frac{\partial}{\partial\phi_3} \right)^T$$

$$\mathbf{g}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\theta}',\boldsymbol{\phi}') := \left. \left[\nabla_{(\boldsymbol{\theta},\boldsymbol{\phi})}\Delta(\boldsymbol{\theta},\boldsymbol{\phi}) \right] \right|_{(\boldsymbol{\theta}',\boldsymbol{\phi}')}$$

$$\mathbf{H}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\theta}',\boldsymbol{\phi}') := \left. \left[\nabla_{(\boldsymbol{\theta},\boldsymbol{\phi})} \left[\nabla_{(\boldsymbol{\theta},\boldsymbol{\phi})}^T \Delta(\boldsymbol{\theta},\boldsymbol{\phi}) \right] \right] \right|_{(\boldsymbol{\theta}',\boldsymbol{\phi}')}.$$

Suppose that, for $\kappa > 0$, $\boldsymbol{\delta y} \in [-\kappa, \kappa]^6$. Then

$$\left| |\Delta(\boldsymbol{\theta}' + \boldsymbol{\delta\theta}', \boldsymbol{\phi}' + \boldsymbol{\delta\phi}')| - |\Delta(\boldsymbol{\theta}', \boldsymbol{\phi}')| \right|$$

$$\leq ||\boldsymbol{\delta y}|| ||\mathbf{g}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\theta}', \boldsymbol{\phi}')|| + \frac{1}{2} ||\boldsymbol{\delta y}||^2 ||\mathbf{H}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\xi}, \boldsymbol{\eta})||$$

for some $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in [\boldsymbol{\theta}' - \kappa\mathbf{1}_{3\times1}, \boldsymbol{\theta}' + \kappa\mathbf{1}_{3\times1}] \times [\boldsymbol{\phi}' - \kappa\mathbf{1}_{3\times1}, \boldsymbol{\phi}' + \kappa\mathbf{1}_{3\times1}]$ where $\mathbf{1}_{3\times1}$ is a 3×1 vector of ones, and $||\cdot||$ is the 2-norm for vectors on $\mathbb{R}^6$ (and the induced 2-norm in the case of the corresponding matrix operators).

Proof
By the mean value theorem (c.f. Section A.6 of [12] for e.g.),

$$\Delta(\boldsymbol{\theta}' + \boldsymbol{\delta\theta}', \boldsymbol{\phi}' + \boldsymbol{\delta\phi}') = \Delta(\boldsymbol{\theta}', \boldsymbol{\phi}') + \boldsymbol{\delta y}^T \mathbf{g}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\theta}', \boldsymbol{\phi}') + \frac{1}{2} \boldsymbol{\delta y}^T \mathbf{H}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\xi}, \boldsymbol{\eta}) \boldsymbol{\delta y} \tag{45}$$

for some $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in [\boldsymbol{\theta}' - \kappa\mathbf{1}_{3\times1}, \boldsymbol{\theta}' + \kappa\mathbf{1}_{3\times1}] \times [\boldsymbol{\phi}' - \kappa\mathbf{1}_{3\times1}, \boldsymbol{\phi}' + \kappa\mathbf{1}_{3\times1}]$.

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Hence

$$\left| |\Delta(\boldsymbol{\theta}' + \boldsymbol{\delta\theta}', \boldsymbol{\phi}' + \boldsymbol{\delta\phi}')| - |\Delta(\boldsymbol{\theta}', \boldsymbol{\phi}')| \right| \leq \left| \Delta(\boldsymbol{\theta}' + \boldsymbol{\delta\theta}', \boldsymbol{\phi}' + \boldsymbol{\delta\phi}') - \Delta(\boldsymbol{\theta}', \boldsymbol{\phi}') \right|$$

$$= \left| \boldsymbol{\delta y}^T \mathbf{g}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\theta}', \boldsymbol{\phi}') + \frac{1}{2} \boldsymbol{\delta y}^T \mathbf{H}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\xi}, \boldsymbol{\eta}) \boldsymbol{\delta y} \right| \leq ||\boldsymbol{\delta y}|| \, ||\mathbf{g}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\theta}', \boldsymbol{\phi}')|| + \frac{1}{2} ||\boldsymbol{\delta y}||^2 ||\mathbf{H}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\xi}, \boldsymbol{\eta})||$$

for some $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in [\boldsymbol{\theta}' - \kappa \mathbf{1}_{3\times 1}, \boldsymbol{\theta}' + \kappa \mathbf{1}_{3\times 1}] \times [\boldsymbol{\phi}' - \kappa \mathbf{1}_{3\times 1}, \boldsymbol{\phi}' + \kappa \mathbf{1}_{3\times 1}]$, where:
the first inequality follows from the reverse triangle inequality; the equality from (45); and the final inequality from the triangle inequality, Cauchy-Schwarz inequality and the compatibility property for matrix norms ([8]). □

We will make full use of Lemma 4 towards the end of the section: however, in the meantime, one should take the hint from this result that bounds on both $||\mathbf{g}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\cdot, \cdot)||$ and $||\mathbf{H}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\cdot, \cdot)||$ will be required.

Proposition 2 ([6]).
(i) for each $n \in \mathcal{I}$,

$$\frac{\partial \Delta}{\partial \theta_n} = \sum_{\pi \in S_3} \frac{\partial \Delta_\pi}{\partial \theta_n}$$

where

$$\frac{\partial \Delta_\pi}{\partial \theta_n} = \begin{cases} -\mathrm{sgn}(\pi) \sin\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & n = \pi(1) \\ \mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \cos\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & n = \pi(2) \\ 0 & n = \pi(3) \end{cases} ;$$

(ii) for each $n \in \mathcal{I}$,

$$\frac{\partial \Delta}{\partial \phi_n} = \sum_{\pi \in S_3} \frac{\partial \Delta_\pi}{\partial \phi_n}$$

where

$$\frac{\partial \Delta_\pi}{\partial \phi_n} = \begin{cases} -\mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \sin\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & n = \pi(1) \\ -\mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \sin\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & n = \pi(2) \\ \mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \cos\left(\phi_{\pi(3)}\right) & n = \pi(3) \end{cases} .$$

Proof
Follows immediately by differentiating (17). □

Proposition 3.
(i) for each $m, n \in \mathcal{I}$,

$$\frac{\partial^2 \Delta}{\partial \theta_m \partial \theta_n} = \sum_{\pi \in S_3} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} = \frac{\partial^2 \Delta}{\partial \theta_n \partial \theta_m}$$

where

$$\frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} = \begin{cases} -\mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & m = n \in \{\pi(1), \pi(2)\} \\ -\mathrm{sgn}(\pi) \sin\left(\theta_{\pi(1)}\right) \cos\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & (m, n) \in \{(\pi(1), \pi(2)), (\pi(2), \pi(1))\} \\ 0 & \pi(3) \in \{m, n\} \end{cases} ;$$

(ii) for each $m, n \in \mathcal{I}$,

$$\frac{\partial^2 \Delta}{\partial \phi_m \partial \phi_n} = \sum_{\pi \in S_3} \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} = \frac{\partial^2 \Delta}{\partial \phi_n \partial \phi_m}$$

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where

$$\frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} = \begin{cases} -\mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & m = n \in \{\pi(1), \pi(2), \pi(3)\} \\ \mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \sin\left(\phi_{\pi(1)}\right) \sin\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & (m,n) \in \{(\pi(1), \pi(2)), (\pi(2), \pi(1))\} \\ -\mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \sin\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \cos\left(\phi_{\pi(3)}\right) & (m,n) \in \{(\pi(1), \pi(3)), (\pi(3), \pi(1))\} \\ -\mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \sin\left(\phi_{\pi(2)}\right) \cos\left(\phi_{\pi(3)}\right) & (m,n) \in \{(\pi(2), \pi(3)), (\pi(3), \pi(2))\} \end{cases} ;$$

(iii) for each $m, n \in \mathcal{I}$,

$$\frac{\partial^2 \Delta}{\partial \theta_m \partial \phi_n} = \sum_{\pi \in S_3} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} = \frac{\partial^2 \Delta}{\partial \phi_n \partial \theta_m}$$

where

$$\frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} = \begin{cases} \mathrm{sgn}(\pi) \sin\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \sin\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & m = \pi(1),\ n = \pi(1) \\ -\mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \cos\left(\theta_{\pi(2)}\right) \sin\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & m = \pi(2),\ n = \pi(1) \\ \mathrm{sgn}(\pi) \sin\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \sin\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & m = \pi(1),\ n = \pi(2) \\ -\mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \cos\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \sin\left(\phi_{\pi(2)}\right) \sin\left(\phi_{\pi(3)}\right) & m = \pi(2),\ n = \pi(2) \\ -\mathrm{sgn}(\pi) \sin\left(\theta_{\pi(1)}\right) \sin\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \cos\left(\phi_{\pi(3)}\right) & m = \pi(1),\ n = \pi(3) \\ \mathrm{sgn}(\pi) \cos\left(\theta_{\pi(1)}\right) \cos\left(\theta_{\pi(2)}\right) \cos\left(\phi_{\pi(1)}\right) \cos\left(\phi_{\pi(2)}\right) \cos\left(\phi_{\pi(3)}\right) & m = \pi(2),\ n = \pi(3) \\ 0 & m = \pi(3) \end{cases} .$$

Proof

The result follows by differentiating the appropriate first partial derivatives given in Proposition 2 and by noting that, on account of the fact that $\Delta(\cdot, \cdot)$ is twice continuously differentiable, one can interchange the order in which the partial derivatives are taken. $\qquad\square$

Lemma 5.

$$(i)\ \left|\frac{\partial \Delta}{\partial \theta_n}\right| \le 4,\ \ n \in \mathcal{I}; \quad (ii)\ \left|\frac{\partial \Delta}{\partial \phi_n}\right| \le 6,\ \ n \in \mathcal{I}.$$

Proof

Observe that

$$\frac{\partial \Delta}{\partial \theta_n} = \sum_{\pi \in S_3} \frac{\partial \Delta_\pi}{\partial \theta_n} = \sum_{\substack{\pi \in S_3 \\ \pi(3) \ne n}} \frac{\partial \Delta_\pi}{\partial \theta_n} + \sum_{\substack{\pi \in S_3 \\ \pi(3) = n}} \frac{\partial \Delta_\pi}{\partial \theta_n}.$$

Working with Proposition 2 part (i) and noting that there are only 4 terms in the first sum on the right hand side, and that the second sum is equal to zero, along with the fact that each of summands is no greater than 1 in absolute value (since they are just finite products of $\sin(\cdot)/\cos(\cdot)$ functions), then

$$\left|\frac{\partial \Delta}{\partial \theta_n}\right| \le \sum_{\substack{\pi \in S_3 \\ \pi(3) \ne n}} \left|\frac{\partial \Delta_\pi}{\partial \theta_n}\right| \le 4.$$

Now observe that

$$\frac{\partial \Delta}{\partial \phi_n} = \sum_{\pi \in S_3} \frac{\partial \Delta_\pi}{\partial \phi_n}$$

Again, working with Proposition 2 part (ii), on account of the fact that each of the summands is no greater than 1 in absolute value, then

$$\left|\frac{\partial \Delta}{\partial \phi_n}\right| \le \sum_{\pi \in S_3} \left|\frac{\partial \Delta_\pi}{\partial \phi_n}\right| \le 6.$$

$\qquad\square$

Lemma 6.

$$(i) \quad \left| \frac{\partial^2 \Delta}{\partial \theta_m \partial \theta_n} \right| \leq \begin{cases} 4 & m = n \\ 2 & m \neq n \end{cases};$$

$$(ii) \quad \left| \frac{\partial^2 \Delta}{\partial \phi_m \partial \phi_n} \right| \leq 6; \quad (iii) \quad \left| \frac{\partial^2 \Delta}{\partial \theta_m \partial \phi_n} \right| \leq 4.$$

Proof

We shall consider each of the three parts, breaking down each one for the cases $m = n$ and $m \neq n$.

$(i); m = n$:

Observe that

$$\frac{\partial^2 \Delta}{\partial \theta_m \partial \theta_n} = \sum_{\pi \in S_3} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} = \sum_{\substack{\pi \in S_3 \\ n = \pi(1)}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} + \sum_{\substack{\pi \in S_3 \\ n = \pi(2)}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} + \sum_{\substack{\pi \in S_3 \\ n = \pi(3)}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n}.$$

Each of the first two sums on the right hand side has exactly 2 terms; whereas, according to Proposition 3 (i), the last sum has no non-zero terms. According to Proposition 3 (i), each of the summands is no greater than 1 in absolute value. Hence

$$\left| \frac{\partial^2 \Delta}{\partial \theta_m \partial \theta_n} \right| \leq \sum_{\substack{\pi \in S_3 \\ n = \pi(1)}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} \right| + \sum_{\substack{\pi \in S_3 \\ n = \pi(2)}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} \right| \leq 2 + 2 = 4.$$

$(i); m \neq n$:

Observe that

$$\frac{\partial^2 \Delta}{\partial \theta_m \partial \theta_n} = \sum_{\pi \in S_3} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} = \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(1),\pi(2)),(\pi(2),\pi(1))\}}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} + \sum_{\substack{\pi \in S_3 \\ \pi(3) \in \{m,n\}}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n}$$

noting that $(m,n) \notin \{(\pi(1), \pi(2)), (\pi(2), \pi(1))\}$ if and only if $\pi(3) \in \{m, n\}$. According to Proposition 3 (i), there are only two terms in the first sum on the right hand side, each of which is no greater than 1 in absolute value, and each of the summands in the final sum on the right hand side is equal to 0. Hence

$$\left| \frac{\partial^2 \Delta}{\partial \theta_m \partial \theta_n} \right| \leq \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(1),\pi(2)),(\pi(2),\pi(1))\}}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} \right| + \sum_{\substack{\pi \in S_3 \\ \pi(3) \in \{m,n\}}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \theta_n} \right| \leq 2 + 0 = 2.$$

$(ii); m = n$:

One can see that

$$\frac{\partial^2 \Delta}{\partial \phi_m \partial \phi_n} = \sum_{\substack{\pi \in S_3 \\ n = \pi(1)}} \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} + \sum_{\substack{\pi \in S_3 \\ n = \pi(2)}} \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} + \sum_{\substack{\pi \in S_3 \\ n = \pi(3)}} \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n}.$$

According to Proposition 3 (ii), each sum on the right hand side consists of two terms and that each of those terms is no greater than 1 in absolute value. Hence

$$\left| \frac{\partial^2 \Delta}{\partial \phi_m \partial \phi_n} \right| \leq \sum_{\substack{\pi \in S_3 \\ n = \pi(1)}} \left| \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} \right| + \sum_{\substack{\pi \in S_3 \\ n = \pi(2)}} \left| \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} \right| + \sum_{\substack{\pi \in S_3 \\ n = \pi(3)}} \left| \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} \right| \leq 2 + 2 + 2 = 6.$$

$(ii); m \neq n$:

One can see that

$$\frac{\partial^2 \Delta}{\partial \phi_m \partial \phi_n} = \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(1),\pi(2)),(\pi(2),\pi(1))\}}} \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n}$$

$$+ \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(1),\pi(3)),(\pi(3),\pi(1))\}}} \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} + \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(2),\pi(3)),(\pi(3),\pi(2))\}}} \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n}.$$

Again, by Proposition 3 (ii), each sum on the right hand side consists of two terms and that each of those terms is no greater than 1 in absolute value. Hence

$$\left| \frac{\partial^2 \Delta}{\partial \phi_m \partial \phi_n} \right| \leq \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(1),\pi(2)),(\pi(2),\pi(1))\}}} \left| \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} \right|$$

$$+ \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(1),\pi(3)),(\pi(3),\pi(1))\}}} \left| \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} \right| + \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(2),\pi(3)),(\pi(3),\pi(2))\}}} \left| \frac{\partial^2 \Delta_\pi}{\partial \phi_m \partial \phi_n} \right| \leq 2 + 2 + 2 = 6.$$

$(iii); m = n$:

One can see that

$$\frac{\partial^2 \Delta}{\partial \theta_m \partial \phi_n} = \sum_{\substack{\pi \in S_3 \\ n=\pi(1)}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} + \sum_{\substack{\pi \in S_3 \\ n=\pi(2)}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} + \sum_{\substack{\pi \in S_3 \\ n=\pi(3)}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n}.$$

According to Proposition 3 (iii), each of the first two sums on the right hand side consists of two terms and that each of the terms is no greater than 1 in absolute value. As for the last sum on the right hand side, then according to Proposition 3 (iii), each of its summands is equal to 0. Hence

$$\left| \frac{\partial^2 \Delta}{\partial \theta_m \partial \phi_n} \right| \leq \sum_{\substack{\pi \in S_3 \\ n=\pi(1)}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} \right| + \sum_{\substack{\pi \in S_3 \\ n=\pi(2)}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} \right| + \sum_{\substack{\pi \in S_3 \\ n=\pi(3)}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} \right| \leq 2 + 2 + 0 = 4.$$

$(iii); m \neq n$:

One can see that

$$\frac{\partial^2 \Delta}{\partial \theta_m \partial \phi_n} = \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(1),\pi(2)),(\pi(2),\pi(1))\}}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n}$$

$$+ \sum_{\substack{\pi \in S_3 \\ (m,n)=(\pi(1),\pi(3))}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} + \sum_{\substack{\pi \in S_3 \\ (m,n)=(\pi(2),\pi(3))}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} + \sum_{\substack{\pi \in S_3 \\ m=\pi(3)}} \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n}.$$

According to Proposition 3 (iii), each summand on the right hand side is no greater than 1 in absolute value. Also notice that the first sum on the right hand side consists of 2 terms, the second just 1 term, the third just 1 term, and that there are no non-zero terms in the final sum. Hence

$$\left| \frac{\partial^2 \Delta}{\partial \theta_m \partial \phi_n} \right| \leq \sum_{\substack{\pi \in S_3 \\ (m,n) \in \{(\pi(1),\pi(2)),(\pi(2),\pi(1))\}}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} \right|$$

$$+ \sum_{\substack{\pi \in S_3 \\ (m,n)=(\pi(1),\pi(3))}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} \right| + \sum_{\substack{\pi \in S_3 \\ (m,n)=(\pi(2),\pi(3))}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} \right| + \sum_{\substack{\pi \in S_3 \\ m=\pi(3)}} \left| \frac{\partial^2 \Delta_\pi}{\partial \theta_m \partial \phi_n} \right|$$

$$\leq 2 + 1 + 1 + 0 = 4.$$

$\square$

Proposition 4.

Suppose that $(\boldsymbol{\delta\theta'}, \boldsymbol{\delta\phi'}) \in [-\kappa, \kappa]^3 \times [-\kappa, \kappa]^3$, where $\kappa > 0$. Then

$$\left| |\Delta(\boldsymbol{\theta'} + \boldsymbol{\delta\theta'}, \boldsymbol{\phi'} + \boldsymbol{\delta\phi'})| - |\Delta(\boldsymbol{\theta'}, \boldsymbol{\phi'})| \right| \leq \text{diff}_{\text{neighbour}}(\kappa)$$

where

$$\text{diff}_{\text{neighbour}}(\kappa) := \kappa\sqrt{936} + \kappa^2\sqrt{6156}.$$

Proof
Setting $\boldsymbol{\delta y}^T := (\boldsymbol{\delta\theta'}^T, \boldsymbol{\delta\phi'}^T)$, then by Lemma 4,

$$\left| |\Delta(\boldsymbol{\theta'} + \boldsymbol{\delta\theta'}, \boldsymbol{\phi'} + \boldsymbol{\delta\phi'})| - |\Delta(\boldsymbol{\theta'}, \boldsymbol{\phi'})| \right|$$

$$\leq ||\boldsymbol{\delta y}|| \, ||\mathbf{g}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\theta'}, \boldsymbol{\phi'})|| + \frac{1}{2}||\boldsymbol{\delta y}||^2 ||\mathbf{H}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\xi}, \boldsymbol{\eta})|| \tag{46}$$

for some $(\boldsymbol{\xi}, \boldsymbol{\eta}) \in [\boldsymbol{\theta'} - \kappa \mathbf{1}_{3\times 1}, \boldsymbol{\theta'} + \kappa \mathbf{1}_{3\times 1}] \times [\boldsymbol{\phi'} - \kappa \mathbf{1}_{3\times 1}, \boldsymbol{\phi'} + \kappa \mathbf{1}_{3\times 1}]$. By the definition of κ, $||\boldsymbol{\delta y}|| \leq \kappa\sqrt{6}$. Denoting the Frobenius norm by $|| \cdot ||_F$, then by Lemma 5,

$$||\mathbf{g}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\theta'}, \boldsymbol{\phi'})||^2 \leq ||\mathbf{g}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\theta'}, \boldsymbol{\phi'})||_F^2$$

$$= \sum_{m \in \{1,2,3\}} \left| \left\{ \frac{\partial \Delta(\boldsymbol{\theta}, \boldsymbol{\phi})}{\partial \theta_m} \right\} \Big|_{(\boldsymbol{\theta'}, \boldsymbol{\phi'})} \right|^2 + \sum_{n \in \{1,2,3\}} \left| \left\{ \frac{\partial \Delta(\boldsymbol{\theta}, \boldsymbol{\phi})}{\partial \phi_n} \right\} \Big|_{(\boldsymbol{\theta'}, \boldsymbol{\phi'})} \right|^2$$

$$\leq (3 \times 4^2) + (3 \times 6^2) = 156.$$

Similarly, but by invoking Lemma 6 instead, then for any $(\boldsymbol{\xi}^*, \boldsymbol{\eta}^*) \in \mathbb{R}^3 \times \mathbb{R}^3$,

$$||\mathbf{H}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\xi}^*, \boldsymbol{\eta}^*)||^2 \leq ||\mathbf{H}_{(\boldsymbol{\theta},\boldsymbol{\phi})}(\boldsymbol{\xi}^*, \boldsymbol{\eta}^*)||_F^2$$

$$= \sum_{m \in \{1,2,3\}} \left| \left\{ \frac{\partial^2 \Delta(\boldsymbol{\theta}, \boldsymbol{\phi})}{\partial \theta_m^2} \right\} \Big|_{(\boldsymbol{\xi}^*, \boldsymbol{\eta}^*)} \right|^2 + \sum_{\substack{(m,n) \in \{1,2,3\}^2 \\ m \neq n}} \left| \left\{ \frac{\partial^2 \Delta(\boldsymbol{\theta}, \boldsymbol{\phi})}{\partial \theta_m \partial \theta_n} \right\} \Big|_{(\boldsymbol{\xi}^*, \boldsymbol{\eta}^*)} \right|^2$$

$$+ \sum_{(m,n) \in \{1,2,3\}^2} \left| \left\{ \frac{\partial^2 \Delta(\boldsymbol{\theta}, \boldsymbol{\phi})}{\partial \phi_m \partial \phi_n} \right\} \Big|_{(\boldsymbol{\xi}^*, \boldsymbol{\eta}^*)} \right|^2 + 2 \sum_{(m,n) \in \{1,2,3\}^2} \left| \left\{ \frac{\partial^2 \Delta(\boldsymbol{\theta}, \boldsymbol{\phi})}{\partial \theta_m \partial \phi_n} \right\} \Big|_{(\boldsymbol{\xi}^*, \boldsymbol{\eta}^*)} \right|^2$$

$$\leq (3 \times 4^2) + (6 \times 2^2) + (9 \times 6^2) + 2(9 \times 4^2) = 684.$$

It now follows that an upper bound on (46) is given by

$$\kappa\sqrt{6}\sqrt{156} + 3\kappa^2\sqrt{684} \quad \text{i.e.} \quad \kappa\sqrt{936} + \kappa^2\sqrt{6156}.$$

$\square$

Next, we introduce functions for rounding any non-negative real number to the nearest integer, and also to m decimal places, respectively.

Definition 6 (rounding functions).
(i) For $x \in [0, \infty)$, define

$$\text{round}(x) := \begin{cases} \lfloor x \rfloor & \text{if} \quad x - \lfloor x \rfloor < 0.5 \\ \lceil x \rceil & \text{if} \quad x - \lfloor x \rfloor \geq 0.5 \end{cases}.$$

(ii) For $y \in [0, \infty)$ and $m \in \mathbb{Z}^+$, define

$$\text{round}(y, m) := \frac{\text{round}(10^m y)}{10^m}.$$

Theorem 4. *Consider a grid search optimization procedure in which:*
(i) $\epsilon := 10^{-n}$ for some $n \in \mathbb{Z}^+$;
(ii) a finite κ_ϵ-net of search points is constructed within the search domain such that no two componentwise adjacent points are spaced out by more than κ_ϵ, where

$$\kappa_\epsilon := \frac{-\sqrt{936} + \sqrt{936 + \epsilon\sqrt{98496}}}{\sqrt{24624}};$$

(iii) the value of $\text{round}(|\Delta(\cdot, \cdot)|, n)$ is available at each point on the κ_ϵ-net.

Then if one chooses the maximum/minimum value of $\text{round}(|\Delta(\cdot, \cdot)|, n)$ on the κ_ϵ-net, call it $\widehat{\Delta}_{\text{opt}}$, then the true maximim/minimum value of $|\Delta(\cdot, \cdot)|$ across the entire search domain, call it Δ_{opt}, will be such that

$$\Delta_{\text{opt}} \in [\widehat{\Delta}_{\text{opt}} - 1.05\epsilon, \ \widehat{\Delta}_{\text{opt}} + 1.05\epsilon).$$

Proof
Consider the equation

$$\text{diff}_{\text{neighbour}}(\kappa) = \epsilon,$$

i.e.

$$\kappa\sqrt{936} + \kappa^2\sqrt{6156} = \epsilon.$$

Then it can be seen that κ_ϵ is the positive root of the above equation. From Proposition 4, it follows that any point within a 6 dimensional hypercube of length $2\kappa_\epsilon$ has a $|\Delta(\cdot, \cdot)|$ value which differs by no more than ϵ from the $|\Delta(\cdot, \cdot)|$ value of the point at the centre of the hypercube. Since $\widehat{\Delta}_{\text{opt}}$ is correct to n decimal places, then it follows that $\widehat{\Delta}_{\text{opt}} = \text{round}(x, n)$, where

$$x \in \left[\widehat{\Delta}_{\text{opt}} - \frac{10^{-(n+1)}}{2}, \widehat{\Delta}_{\text{opt}} + \frac{10^{-(n+1)}}{2}\right).$$

Hence

$$\Delta_{\text{opt}} \in \left[\widehat{\Delta}_{\text{opt}} - \frac{10^{-(n+1)}}{2} - \epsilon, \widehat{\Delta}_{\text{opt}} + \frac{10^{-(n+1)}}{2} + \epsilon\right)$$

i.e.

$$\Delta_{\text{opt}} \in \left[\widehat{\Delta}_{\text{opt}} - \frac{10^{-(n+1)}}{2} - 10^{-n}, \widehat{\Delta}_{\text{opt}} + \frac{10^{-(n+1)}}{2} + 10^{-n}\right).$$

However

$$\frac{10^{-(n+1)}}{2} + 10^{-n} = \left(1 + \frac{1}{20}\right)10^{-n} = 1.05\epsilon.$$

$\square$

5 Numerical Examples

Unless specified to the contrary, speeds are expressed in metres per second, and angles in radians.

Example 1: fixed demanded angles - large $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$

$$\widetilde{\mathbf{v}}_0^T = (16.12, -16.73, 0.43);$$

$$\boldsymbol{\theta}_0^T = (0, 2\pi/3, 4\pi/3) = (0, 2.0944, 4.1888);$$

$$\boldsymbol{\phi}_0^T = (0.7834, 0.7834, 0.7834);$$

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (16.07, -16.78, 0.38); \quad \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (16.17, -16.68, 0.48);$$

$$\text{vol}\left(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right) = 0.001;$$

The above provides the numerical values for the nominal input parameters, $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, and indicates that the uncertainty around each component of the Doppler velocity vector is ± 0.05 metres per second. The value of $|\Delta(\cdot, \cdot)|$ at the nominal angle settings, as well as that of the nominal reconstructed velocity, can also be computed.

$$|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)| = 0.9204.$$

Reconstructed velocity $\mathbf{v}_0 = \mathbf{M}^{-1}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\widetilde{\mathbf{v}}_0$	**(22.8365, -13.9830, -0.0851)**
vol $\left(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right)$	**0.0011**

A plot depicting the value of the nominal reconstructed velocity vector, and in relation to the region of output uncertainty, is presented in Figure 2.
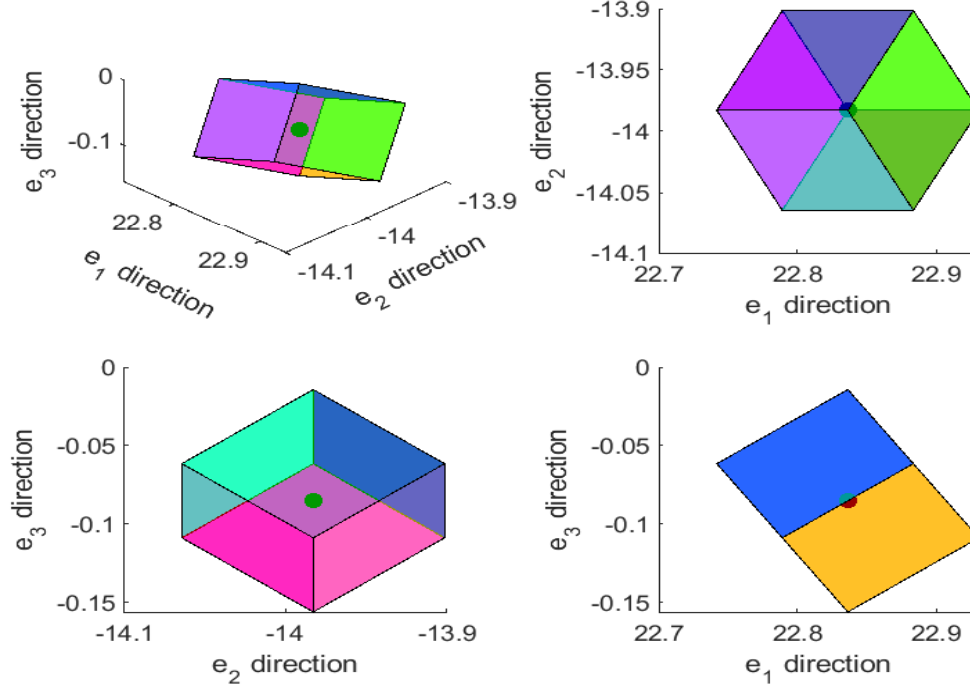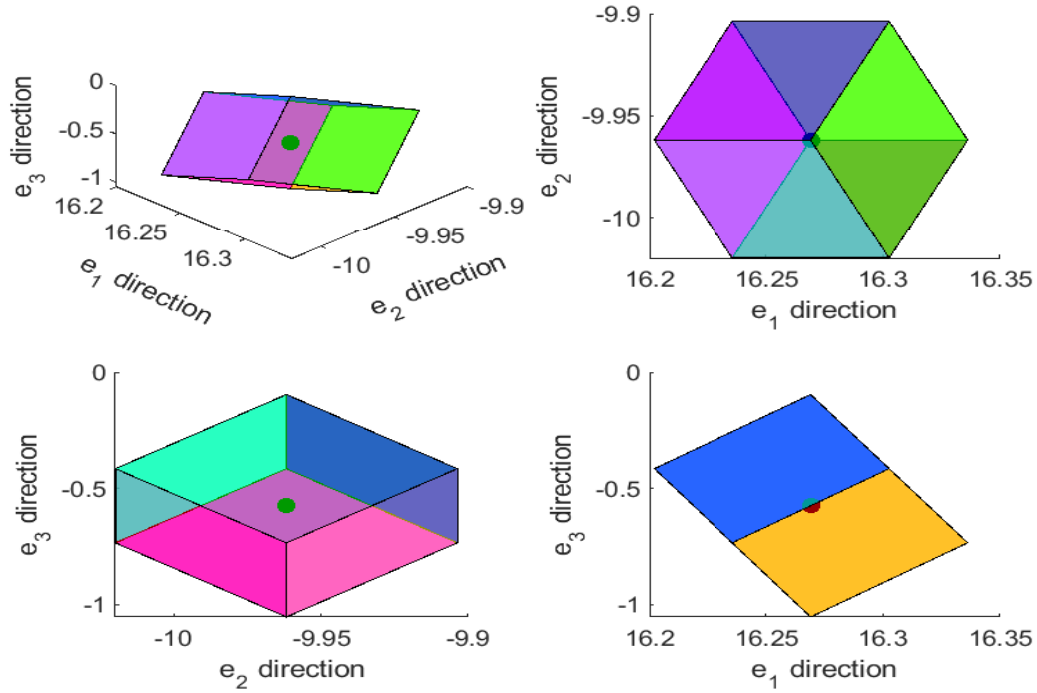
Figure 2: Plots of reconstructed velocity (indicated by the dot) in terms of Cartesian co-ordinates, embedded within the region of uncertainty (parellepiped), from general viewpoint [top left] along with orthographic projections, for a large $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$ scenario.

Example 2: fixed demanded angles - small $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$

A similar exercise to the one of Example 1 is carried out here, except that this time the elevation angles are chosen to be substantially lower, to the extent that the value of $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$ is rendered to be small.

$$\widetilde{\mathbf{v}}_0^T = (16.12, -16.73, 0.43);$$

$$\boldsymbol{\theta}_0^T = (0, 2\pi/3, 4\pi/3) = (0, 2.0944, 4.1888);$$

$$\boldsymbol{\phi}_0^T = (\pi/30, \pi/30, \pi/30) = (0.1047, 0.1047, 0.1047);$$

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (16.07, -16.78, 0.38); \quad \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (16.17, -16.68, 0.48);$$

$$\mathrm{vol}\left(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right) = 0.001;$$

$$|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)| = 0.2686.$$

Reconstructed velocity $\mathbf{v}_0 = \mathbf{M}^{-1}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\widetilde{\mathbf{v}}_0$	**(16.2691, -9.9617, -0.5746)**
$\mathrm{vol}\left(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right)$	**0.0037**

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Figure 3: Plots of reconstructed velocity (indicated by the dot) in terms of Cartesian co-ordinates, embedded within the region of uncertainty (parellepiped), from general viewpoint [top left] along with orthographic projections, for a small $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$ scenario.

Example 3: uncertain demanded angles - large $|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)|$

In contradistinction to the previous examples, here we additionally allow for uncertainty in relation to both the azimuthal and elevation angles.

$$\widetilde{\mathbf{v}}_0^T = (16.12, -16.73, 0.43);$$

$$\boldsymbol{\theta}_0^T = (0, 2\pi/3, 4\pi/3) = (0, 2.0944, 4.1888);$$

$$\boldsymbol{\phi}_0^T = (\pi/4, \pi/4, \pi/4) = (0.7854, 0.7854, 0.7854);$$

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (16.07, -16.78, 0.38); \quad \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (16.17, -16.68, 0.48);$$

$$\boldsymbol{\theta}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (6.2822, 2.0934, 4.1878); \quad \boldsymbol{\theta}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (0.0010, 2.0954, 4.1898);$$

$$\boldsymbol{\phi}^{\min}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (0.7844, 0.7844, 0.7844); \quad \boldsymbol{\phi}^{\max}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)^T = (0.7864, 0.7864, 0.7864);$$

$$\text{vol}\left(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right) = 0.001;$$

$$|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)| = 0.9186.$$

Reconstructed velocity $\mathbf{v}_0 = \mathbf{M}^{-1}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\widetilde{\mathbf{v}}_0$	(22.8822, -14.0109, -0.0849)

We next consider the calculation of bounds for the minimum/maximum value of $|\Delta(\cdot,\cdot)|$ across $R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)$. In order to ascertain the spacing along each of the 6 dimensions of an appropriate $\kappa_{0.01}$-net, we evaluate the formula for κ_ϵ given in Theorem 4, with $\epsilon = 0.01$: this yields a value for $\kappa_{0.01}$ equal to 0.00032658669. Therefore the number of sections (assumed to be of equal length, for simplicity) separated by points on the $\kappa_{0.01}$-net along each dimension should be at least

$$0.002/0.00032658669 = 6.1239482846 \approx 7$$

in order to satisfy the requirement of locating the optimum to within ± 0.0105. In our particular implementation of the grid search optimization, the section lengths will be set equal to 0.0002, corresponding to 10 sections (and therefore 11 points) along each dimension.

The smallest value of $|\Delta(\cdot,\cdot)|$ on the $\kappa_{0.01}$-net was found to be 0.9176 (to 4 d.p.) and thus, according to Theorem 4, it follows that

$$0.9095 = 0.92 - 0.0105 \leq \min_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta},\boldsymbol{\phi})| \leq 0.92 + 0.0105 = 0.9305.$$

The largest value of $|\Delta(\cdot,\cdot)|$ on the $\kappa_{0.01}$-net was found to be 0.9195 (to 4 d.p.). Again, according to Theorem 4, it follows that

$$0.9095 = 0.92 - 0.0105 \leq \max_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta},\boldsymbol{\phi})| \leq 0.92 + 0.0105 = 0.9305.$$

Recall from Theorem 2 that

$$\text{vol}\left(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\right) \geq \frac{\text{vol}\left(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\right)}{\max\left\{|\Delta(\boldsymbol{\theta}'',\boldsymbol{\phi}'')| : (\boldsymbol{\theta}'',\boldsymbol{\phi}'') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\right\}}$$

and that from Theorem 3

$$\text{vol}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)) \begin{cases} \geq & \frac{\text{vol}(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0))}{\max\left\{|\Delta(\boldsymbol{\theta}'',\boldsymbol{\phi}'')|:(\boldsymbol{\theta}'',\boldsymbol{\phi}'')\in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\right\}} \\ \leq & \frac{\text{vol}(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0))}{\min\left\{|\Delta(\boldsymbol{\theta}'',\boldsymbol{\phi}'')|:(\boldsymbol{\theta}'',\boldsymbol{\phi}'')\in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\right\}} \end{cases}.$$

If one uses 0.9305 rather than $\max_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta},\boldsymbol{\phi})|$ in the denominator of the lower bound, and 0.9095 rather than $\min_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)\times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta},\boldsymbol{\phi})|$ in the denominator of the upper bound, then it follows that, to 4 s.f.,

$$\left\| \frac{\text{vol}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0))}{\text{vol}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0))} \right\| \geq \left\| 0.001 \times \frac{1}{0.9305} = 0.001 \times 1.07469102633 = \mathbf{0.001075} \right\|$$

and

$$\left\| \text{vol}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)) \right\| \leq \left\| 0.001 \times \frac{1}{0.9095} = 0.001 \times 1.09950522265 = \mathbf{0.001100} \right\|$$
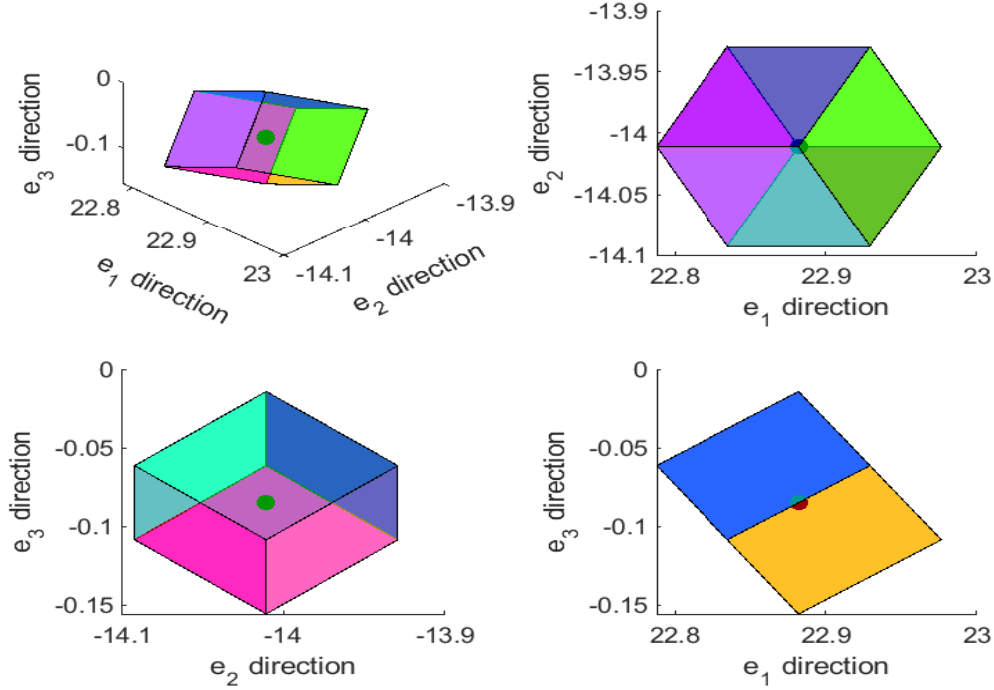
Figure 4: Plots of reconstructed velocity (indicated by the dot) in terms of Cartesian co-ordinates, embedded within the "nominal" region of uncertainty (parellepiped), $R_{\mathbf{v}^*|(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0)}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)$, from general viewpoint [top left] along with orthographic projections, for a large $|\Delta(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0)|$ scenario.

In essence, therefore, the lack of perfect orthogonality in the LIDAR beam orientations leads to an inflation of the volumetric uncertainty brought about by the Doppler velocity measurements of between 7% and 10%.

Example 4: uncertain demanded angles - small $|\Delta(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0)|$

The calculations for this example follow along similar lines as those for the previous one, except that the nominal elevation angles (and hence the associated angle ranges) are taken to be much smaller, so as to render $|\Delta(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0)|$ and the minimum/maximum value of $|\Delta(\cdot,\cdot)|$ across $R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)$ to be small.

$$\widetilde{\mathbf{v}}_0^T = (16.12, -16.73, 0.43);$$

$$\boldsymbol{\theta}_0^T = (0, 2\pi/3, 4\pi/3) = (0, 2.0944, 4.1888);$$

$$\boldsymbol{\phi}_0^T = (\pi/60, \pi/60, \pi/60) = (0.0524, 0.0524, 0.0524);$$

$$\widetilde{\mathbf{v}}^{\min}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)^T = (16.07, -16.78, 0.38); \quad \widetilde{\mathbf{v}}^{\max}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)^T = (16.17, -16.68, 0.48);$$

$$\boldsymbol{\theta}^{\min}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)^T = (6.2822, 2.0934, 4.1878); \quad \boldsymbol{\theta}^{\max}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)^T = (0.0010, 2.0954, 4.1898);$$

$$\boldsymbol{\phi}^{\min}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)^T = (0.0514, 0.0514, 0.0514); \quad \boldsymbol{\phi}^{\max}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)^T = (0.0534, 0.0534, 0.0534);$$

$$\text{vol}\left(R_{\widetilde{\mathbf{v}}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)\right) = 0.001;$$

$$|\Delta(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)| = 0.1357.$$

Reconstructed velocity $\mathbf{v}_0 = \mathbf{M}^{-1}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)\widetilde{\mathbf{v}}_0$	(16.2024, -9.9208, -1.1455)

The smallest value of $|\Delta(\cdot, \cdot)|$ on the $\kappa_{0.01}$-net was found to be 0.1331 (to 4 d.p.) and thus, according to Theorem 4, it follows that

$$0.1195 = 0.13 - 0.0105 \le \min_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta}, \boldsymbol{\phi})| \le 0.13 + 0.0105 = 0.1405.$$

The largest value of $|\Delta(\cdot, \cdot)|$ on the $\kappa_{0.01}$-net was found to be 0.1383 (to 4 d.p.). Again, according to Theorem 4, it follows that

$$0.1295 = 0.14 - 0.0105 \le \max_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta}, \boldsymbol{\phi})| \le 0.14 + 0.0105 = 0.1505.$$

Using 0.1505 rather than $\displaystyle\max_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta}, \boldsymbol{\phi})|$

and 0.1195 rather than $\displaystyle\min_{R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)} |\Delta(\boldsymbol{\theta}, \boldsymbol{\phi})|$, then it follows that, to 4 s.f.,

$\dfrac{\text{vol}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))}{\text{vol}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))}$	$\ge$	$0.001 \times \frac{1}{0.1505} = 0.001 \times 6.64451827243 = \mathbf{0.006645}$

and

$\text{vol}_{\text{Av}}(R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0))$	$\le$	$0.001 \times \frac{1}{0.1195} = 0.001 \times 8.36820083682 = \mathbf{0.008368}$

In this case, the distinct lack of orthogonality in the LIDAR beam orientations leads to an inflation of the volumetric uncertainty brought about by the Doppler velocity measurements of somewhere between 500% and 800%: this only further confirms that configurations corresponding to low values of $|\Delta(\cdot, \cdot)|$ should be avoided.
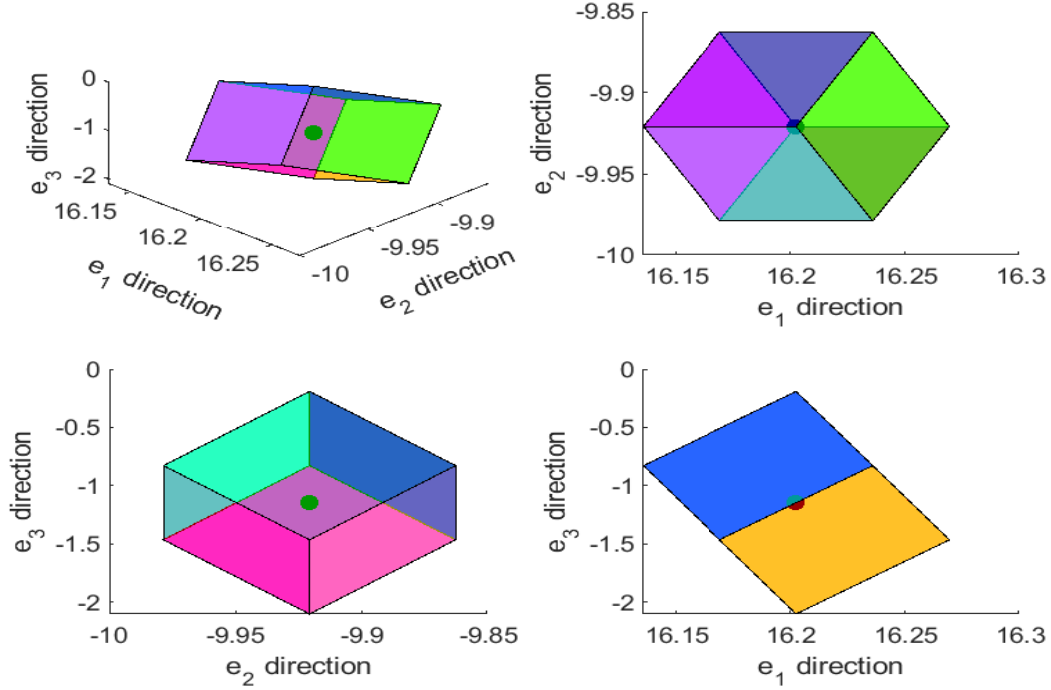
Figure 5: Plots of reconstructed velocity (indicated by the dot) in terms of Cartesian co-ordinates, embedded within the "nominal" region of uncertainty (parellepiped), $R_{\mathbf{v}^*|(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0)}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)$, from general viewpoint [top left] along with orthographic projections, for a small $|\Delta(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0)|$ scenario.

6   Summary of Main Results and Concluding Remarks

For a given nominal input parameter setting $(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)$, we have presented a characterization, and quantification of the size of, the set of points, $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)$, occupied by the reconstructed velocity vector, $\mathbf{v}^*$, when the uncertainty around either certain, or all, of the input parameters has compact interval support (and, additionally, where necessary, its components are independent and uniformly distributed). We summarize below, under the italicized headings, the main results that have been established.

No uncertainty around the demanded angle configurations, $(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0)$, but uniform uncertainty around the Doppler velocity readings, $\widetilde{\mathbf{v}}_0$, that appear on the "LIDAR computer":
- $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)$ takes the form of a parallelepiped;
- the distribution of $\mathbf{v}^*$ adheres to Lebesgue measure within that set;
- the volume of $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)$ is given by the volume of the set occupied by $\widetilde{\mathbf{v}}^*$ divided through by the volume of the parallelepiped whose edges are of unit length and whose orientations are governed by $(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0)$.

Uniform uncertainty around both the demanded angle configurations, $(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0)$, and Doppler velocity readings, $\widetilde{\mathbf{v}}_0$, that appear on the "LIDAR computer":
- the conditional range of $\mathbf{v}^*$ given $(\boldsymbol{\theta}',\boldsymbol{\phi}') \in R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0,\boldsymbol{\phi}_0,\widetilde{\mathbf{v}}_0)$, namely

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$R_{\mathbf{v}^*|(\boldsymbol{\theta}',\boldsymbol{\phi}')}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, takes the form of a parallelepiped;
- a lower bound on the volume of $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ is given by the volume of the set occupied by $\widetilde{\mathbf{v}}^*$ divided through by the maximized volume, over the angle uncertainty range that is parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$, of the parallelepiped whose edges comprise the Doppler unit vectors;
- a lower bound on the "averaged volume" of $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ (with respect to the angle uncertainty range parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$) is given by the volume of the set occupied by $\widetilde{\mathbf{v}}^*$ divided through by the maximized volume, over the angle uncertainty range that is parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$, of the parallelepiped whose edges comprise the Doppler unit vectors;
- an upper bound on the "averaged volume" of $R_{\mathbf{v}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$ (with respect to the angle uncertainty range parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$) is given by the volume of the set occupied by $\widetilde{\mathbf{v}}^*$ divided through by the minimized volume, over the angle uncertainty range that is parameterized by $(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0)$, of the parallelepiped whose edges comprise the Doppler unit vectors.

Grid Search Optimization Algorithm for maximimization/minimization of $|\Delta(\cdot, \cdot)|$ over $R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$:
- bounds are provided on the gradient function of $\Delta(\cdot, \cdot)$;
- bounds are provided on the Hessian function of $\Delta(\cdot, \cdot)$;
- for a given $\epsilon = 10^{-n}$, $n \in \mathbb{Z}^+$, a procedure is presented that generates bounds on Δ_{opt}, i.e. the maximum/minimum of $|\Delta(\cdot, \cdot)|$ over $R_{\boldsymbol{\theta}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0) \times R_{\boldsymbol{\phi}^*}(\boldsymbol{\theta}_0, \boldsymbol{\phi}_0, \widetilde{\mathbf{v}}_0)$, such that Δ_{opt} is within $\pm 1.05\epsilon$ of the maximum/minimum value of $|\Delta(\cdot, \cdot)|$ calculated, to n decimal places, across a specially constructed search grid of points.

LIDAR placement policy advice:
From the structure of the volumetric bounds that have been derived in this paper, the overall conclusion and practitioner advice can be summarized as follows:-
- to ensure that the smallest achievable reconstructed volume (whether actual, or averaged) is kept to a minimum, it is suggested that one chooses nominal LIDAR orientations that are either mutually orthogonal, or else for which $|\Delta(\cdot, \cdot)|$ is as large as possible;
- to ensure that the largest achievable reconstructed volume (whether actual, or averaged) is kept to a minimum, it is suggested that one chooses nominal LIDAR orientations that are either mutually orthogonal, or else for which $|\Delta(\cdot, \cdot)|$ is as large as possible.

Acknowledgements

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